Cowles Foundation Paper 560

Biometrika (1982), **69**, 1, *pp.* 261–4
Printed in Great Britain

The true characteristic function of the F distribution

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SUMMARY

Formulae that are given in the literature for the characteristic function of the F distribution are incorrect and imply that the distribution has finite moments of all orders. Correct formulae are derived and the asymptotic behaviour of the characteristic function in the neighbourhood of the origin is characterized.

Some key words: Asymptotic series; Confluent hypergeometric function; Moment; Tail probability expansion.

1. INTRODUCTION

Major reference works on distribution theory such as Johnson & Kotz (1970, p. 78) state that the characteristic function of an F variate is given by the confluent hypergeometric function in the central case and an infinite series of confluent hypergeometric functions in the noncentral case. Thus, if F has a central F distribution with n_1 and n_2 degrees of freedom the characteristic function of F is stated to be

$$\phi(s) = E\{\exp(iFs)\} = {}_1F_1(\tfrac{1}{2}n_1, -\tfrac{1}{2}n_2; -n_1^{-1} n_2 is), \tag{1}$$

where ${}_1F_1(\,.\,,\,.\,;\,.\,)$ is the confluent hypergeometric function. In the noncentral case, the characteristic function is said to be given by the series

$$\phi(s) = e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\lambda)^j}{j!}\, {}_1F_1(\tfrac{1}{2}n_1 + j, -\tfrac{1}{2}n_2; -n_1^{-1} n_2 is), \tag{2}$$

where λ is the noncentrality parameter; see Patnaik (1949, p. 221) and Johnson & Kotz (1970, p. 190).

The fact that expressions (1) and (2) are incorrect seems to have passed unnoticed in the literature. To see that these expressions are wrong we need only observe that, whereas the F distribution has finite moments of order less than $\frac{1}{2}n_2$, equations (1) and (2) imply that all moments of the distribution are finite since ${}_1F_1(a, b; z)$ is an entire function of z. The problem is that (1) and (2) are close to, but do not represent exactly, only one part of the relevant characteristic functions. As we see in the next section, the remaining parts do not possess continuous derivatives of all orders at the origin. These nonanalytic components are omitted completely from (1) and (2).

2. FORM OF THE CHARACTERISTIC FUNCTION

We start with a central F variate with n_1 and n_2 degrees of freedom and density function

$$p(x) = \frac{n_1^{\frac{1}{2}n_1} n_2^{\frac{1}{2}n_2}}{B(\frac{1}{2}n_1, \frac{1}{2}n_2)} \frac{x^{\frac{1}{2}n_1 - 1}}{(n_2 + n_1 x)^{\frac{1}{2}(n_1 + n_2)}} \quad (x > 0). \tag{3}$$

The characteristic function of F is then given by the integral

$$\phi(s) = \frac{1}{B(\frac{1}{2}n_1, \frac{1}{2}n_2)} \int_0^\infty \exp(isn_1^{-1} n_2 y)\, y^{\frac{1}{2}n_1 - 1}(1+y)^{-\frac{1}{2}(n_1+n_2)}\, dy. \tag{4}$$

For the complex variable z and complex parameters a and c such that $\operatorname{Re}(z) > 0$ and $\operatorname{Re}(a) > 0$, we have that

$$\int_0^\infty e^{-zt} t^{a-1}(1+t)^{c-a-1}\, dt = \Gamma(a)\Psi(a, c; z), \tag{5}$$

which defines the confluent hypergeometric function of the second kind; see Erdéyli (1953, p. 255) and Lebedev (1972, pp. 267–8). Now the domain of definition of the function $\Psi(a, c; z)$ can be extended beyond $\operatorname{Re}(z) > 0$. In the present case, we note that the integral representation continues to hold for z on the imaginary axis if $\operatorname{Re}\{(c-a-1)+(a-1)\} < -1$. That is, if $\operatorname{Re}(c) \leqslant 1-\varepsilon$ for some $\varepsilon > 0$, since this condition ensures that the integral converges absolutely.

It now follows from (4) and (5) that

$$\begin{aligned}\phi(s) &= \frac{\Gamma(\frac{1}{2}n_1)}{B(\frac{1}{2}n_1, \frac{1}{2}n_2)} \Psi(\tfrac{1}{2}n_1, 1-\tfrac{1}{2}n_2;\ -n_1^{-1} n_2 is) \\ &= \frac{\Gamma(\frac{1}{2}n_1 + \frac{1}{2}n_2)}{\Gamma(\frac{1}{2}n_2)} \Psi(\tfrac{1}{2}n_1, 1-\tfrac{1}{2}n_2;\ -n_1^{-1} n_2 is).\end{aligned} \tag{6}$$

Series representations of (6) can be obtained from the following formulae given by Erdéyli (1953, pp. 257, 261):

$$\Psi(a, c; z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\, {}_1F_1(a, c; z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c}\, {}_1F_1(a-c+1, 2-c; z) \tag{7}$$

for nonintegral c; and, for $c = 1-n$ with $n = 0, 1, 2, \ldots$,

$$\begin{aligned}\Psi(a, 1-n; z) &= z^n \Psi(a+n, n+1; z) \\ &= \frac{(-1)^{n-1} z^n}{n!\,\Gamma(a)} \Bigg[{}_1F_1(a+n, n+1; z)\log z + \sum_{r=0}^\infty \frac{(a+n)_r}{(n+1)_r\, r!} \\ &\quad \times \{\psi(a+n+r) - \psi(1+r) - \psi(1+n+r)\} z^r \Bigg] + \frac{(n-1)!}{\Gamma(a+n)} \sum_{r=0}^{n-1} \frac{(a)_r}{(1-n)_r\, r!} z^r,\end{aligned} \tag{8}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function and the final term in (8) is omitted if $n = 0$.

Setting $a = \frac{1}{2}n_1$, and $c = 1-\frac{1}{2}n_2$ in (7), we deduce, for the case of n_2 odd, from (7) that $\phi(s)$ has continuous derivatives at the origin up to order M, where M is the largest integer $< \frac{1}{2}n_2$. Since

$$\frac{d^n}{dz^n}\, {}_1F_1(a, c; z) = \frac{(a)_n}{(c)_n}\, {}_1F_1(a+n, c+n; z),$$

we find from (6) and (7) that

$$\begin{aligned} i^{-j}\phi^{(j)}(0) &= \frac{(\frac{1}{2}n_1)_j}{(1-\frac{1}{2}n_2)_j}\left(-\frac{n_2}{n_1}\right)^j \\ &= \left(\frac{n_2}{n_1}\right)^j \frac{n_1(n_1+2)\ldots(n_1+2j-2)}{(n_2-2)(n_2-4)\ldots(n_2-2j)} \quad (j = 1, \ldots, M),\end{aligned}$$

which corresponds to the known jth moment about the origin of the F distribution. Setting $a = \frac{1}{2}n_1$ and $c = 1-\frac{1}{2}n_2 = 1-n$ in (8) we deduce also that, for the case of n_2 even, $\phi(s)$ has continuous derivatives at the origin up to order $M = n-1$. These derivatives can be obtained from the coefficients in the final sum of (8).

For nonintegral $c = 1-\frac{1}{2}n_2$, we note from (7) that the first term in the representation of function $\Psi(a, c; z)$ in terms of the confluent hypergeometric function, is close to but not equivalent to the expression (1) for the characteristic function given in the literature. Specifically, for $a = \frac{1}{2}n_1$, $c = 1-\frac{1}{2}n_2$ for n_2 odd and $z = -(n_2/n_1)is$, we have

$$\frac{\Gamma(\frac{1}{2}n_1+\frac{1}{2}n_2)}{\Gamma(\frac{1}{2}n_2)}\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,{}_1F_1(a, c; z) = {}_1F_1(\tfrac{1}{2}n_1, 1-\tfrac{1}{2}n_2;\ -n_1^{-1}\,n_2\,is).$$

Note also that, for n_2 even, the conventional expression (1) for the characteristic function is undefined since the confluent hypergeometric function ${}_1F_1(a, c; z)$ has simple poles at the points $c = 0, -1, -2, \ldots$.

These results can readily be extended to the case of a noncentral F variate and we only state the final formula here. If F has a noncentral F distribution with degrees of freedom n_1 and n_2 and noncentrality parameter λ, then the characteristic function of F is

$$\phi(s) = \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{1}{2}n_2)}\sum_{j=0}^{\infty}\frac{(\frac{1}{2}\lambda)^j\,\Gamma(\frac{1}{2}n_1+\frac{1}{2}n_2+j)}{j!}\Psi(\tfrac{1}{2}n_1+j, 1-\tfrac{1}{2}n_2;\ -n_1^{-1}\,n_2\,is), \tag{9}$$

where, as before, the series representation (7) and (8) for the $\Psi(\,.\,,\,.\,;\,.\,)$ function can be used in the respective cases of n_2 odd and n_2 even.

3. Behaviour of the characteristic function as $s \to 0$

When n_2 is odd, we have the following expansion directly from (6) and (7):

$$\phi(s) = \sum_{k=0}^{\infty}\frac{(\frac{1}{2}n_1)_k(-n_2/n_1)^k}{(1-\frac{1}{2}n_2)_k\,k!}(is)^k + \frac{\Gamma(-\frac{1}{2}n_2)}{B(\frac{1}{2}n_1, \frac{1}{2}n_2)}|s|^{\frac{1}{2}n_2}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}n_1+\frac{1}{2}n_2)_m(\frac{1}{4}in_2\pi)^l(-n_2/n_1)^m}{(1+\frac{1}{2}n_2)_m\,l!\,m!}\{\operatorname{sgn}(s)\}^l(is)^m.$$

In particular, this series applies as $s \to 0$. A related formula, which is further complicated by terms involving $\log|s|$, applies in the case where n_2 is even. In fact, the F distribution belongs to the class of characteristic functions whose behaviour as $s \to 0$ is given by the asymptotic series

$$\phi(s) \sim e^{i\eta s}\left[\sum_{m=0}^{M-1}p_m(is)^{m+}|s|^{\mu}\sum_{j=0}^{\infty}\sum_{k=0}^{K(j)}\sum_{l=0}^{L(j)}q_{jkl}|s|^{\nu j}\{i\operatorname{sgn}(s)\}^k(\log|s|)^l\right], \tag{10}$$

where p_m and q_{jkl} are coefficients, $\mu \geqslant M$, $v > 0$ and $L(j)$ is usually either 0 or 1 for all j, as it is for the F distribution. The first sum in (10) is analytic and ensures that integral moments of the distribution exist to order $M-1$ if this is an even integer and $M-2$ if $M-1$ is odd. The second sum in (10) is instrumental in determining the form of the tails of the distribution corresponding to $\phi(s)$. When the density function $p(x)$ exists, (10) can be used to extract an asymptotic expansion of $p(x)$ as $|x| \to \pm\infty$. This work is detailed by Phillips (1981).

The research reported in this paper was supported by the National Science Foundation.

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[*Received July* 1981. *Revised August* 1981]