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A SIMPLE PROOF OF THE LATENT ROOT SENSITIVITY FORMULA *

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This note provides a simple proof of the formula for the partial derivatives of the latent roots of a matrix with respect to its coefficients.

In the analysis of systems of dynamic equations it is often useful to measure the sensitivity of the latent roots with respect to different elements of the matrix of underlying coefficients in the companion form. This is usually achieved by taking partial derivatives of the latent roots with respect to the appropriate coefficients. These partial derivatives serve many functions, an important example of which in statistical analysis arises in the computation of asymptotic standard errors for the latent roots in an estimated dynamic system. Other applications occur in comparative dynamics and in considering the numerical stability of the eigenvalue problem.

The sensitivity formula given in (1) below for these partial derivatives is well known and published proofs have appeared in Laughton (1964) and recently in Gandolfo (1981, pp. 31–37). These proofs are very lengthy and the following simple derivation of the result seems not to be well known and may be of interest.

Lemma. *If the* $n \times n$ *matrix* $A = (a_{ij})$ *has distinct latent roots* $\lambda_1, \ldots, \lambda_n$ *and* T *is a non-singular matrix for which* $T^{-1}AT = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

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then

$$\partial\lambda_m/\partial a_{ij} = \left(T^{-1}E_{ij}T\right)_{m,m}, \tag{1}$$

where E_{ij} has unity in the (i,j) position and zeroes elsewhere.

Proof. We write A in canonical form as $T\Lambda T^{-1}$ and taking differentials we have

$$\mathrm{d}A = \mathrm{d}T\Lambda T^{-1} + T\,\mathrm{d}\Lambda T^{-1} - T\Lambda T^{-1}\,\mathrm{d}TT^{-1} \tag{2}$$

or equivalently

$$T^{-1}\,\mathrm{d}AT = T^{-1}\,\mathrm{d}T\Lambda - \Lambda T^{-1}\,\mathrm{d}T + \mathrm{d}\Lambda.$$

We now note that the diagonal elements of $T^{-1}\,\mathrm{d}T\Lambda$ and $\Lambda T^{-1}\,\mathrm{d}T$ are identical so that $\mathrm{d}\lambda_m = (T^{-1}\,\mathrm{d}AT)_{m,m}$ and we have the partial derivative

$$\partial\lambda_m/\partial a_{ij} = \left(T^{-1}E_{ij}T\right)_{m,m}.$$

This is just the (i,j)th element of the outer product of the latent row and column vectors of λ_m. Q.E.D.

Eq. (2) above gives rise to an alternative formula for (1) which sometimes occurs in numerical linear algebra. We let $*$ denote the complex conjugate transpose of a matrix and let S be a non-singular matrix for which $SA^*S^{-1} = \bar{\Lambda}$. It follows from (2) that

$$S^{*-1}\,\mathrm{d}AT = S^{*-1}\,\mathrm{d}T\Lambda + S^{*-1}T\,\mathrm{d}\Lambda - S^{*-1}A\,\mathrm{d}T.$$

Noting that $S^{*-1}\,\mathrm{d}T\Lambda$, $\Lambda S^{*-1}\,\mathrm{d}T$ and, hence $S^{*-1}A\,\mathrm{d}T$ have the same diagonal elements we deduce that

$$\mathrm{d}\lambda_m = (S^{*-1}\,\mathrm{d}AT)_{m,m}/(S^{*-1}T)_{m,m}, \tag{3}$$

a formula derived by Faddeev and Faddeeva (1963). Of course, (3) reduces to (2) when we write $S^* = TP$ for some non-singular diagonal matrix P.

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