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THE EXACT DISTRIBUTION OF INSTRUMENTAL VARIABLE ESTIMATORS IN AN EQUATION CONTAINING $n+1$ ENDOGENOUS VARIABLES

BY P. C. B. PHILLIPS[1]

This paper derives the exact probability density function of instrumental variable estimators of the coefficient vector of the endogenous variables in a structural equation containing $n+1$ endogenous variables and N degrees of overidentification. This generalizes the presently known results for the special cases where $n = 1$ or 2 and $N = 0$. The usual classical assumptions [**19**] are made of nonrandom exogenous variables and normally distributed disturbances. Some numerical computations are reported for the case $n = 2$.

1. INTRODUCTION

IN THE LATE 1960's, Richardson [**18**] and Sawa [**20**] derived the exact distribution of the two-stage least squares (2SLS) estimator in a structural equation (of a simultaneous system) that contained two endogenous variables and an arbitrary number of degrees of overidentification. Their results refer to the 2SLS estimator of the coefficient of the endogenous variable included on the right hand side of the equation and were obtained under the classical assumptions (to use the term employed by Sargan [**19**]) of normally distributed disturbances and nonrandom exogenous variables.

Very little exact finite sample theory has been published so far for estimators in structural equations containing more than two endogenous variables. Basmann et al. [**4**] extract the joint probability density function (p.d.f.) of the 2SLS estimator in a just identified equation containing three endogenous variables. Basmann [**3**] quotes a result due to Richardson for the same set up but with an arbitrary number of degrees of overidentification[2]. In Basmann's notation, this last result characterizes the subclass

$$\bigcup_{N=1}^{\infty} H_{2,N}$$

where $H_{n,N}$ denotes the joint distribution on $\mathbb{R}^n$ of the 2SLS estimators of the coefficients of the n right-hand side endogenous variables in an equation with N degrees of overidentification. More recently, Sargan [**19**] (Appendix B) has

[1] I am grateful to the referees for their helpful comments on the original version. One referee has very kindly brought my attention to the recent work by Davis [**7, 8**] on invariant polynomials of two matrix arguments. These polynomials extend the zonal polynomials of James [**12**] and provide an alternative means of reducing the $n(n+1)/2$ dimensional integral that is the main obstacle in extracting the exact density of the instrumental variable estimator dealt with in the present paper. The alternative approach and the alternative form of the joint density function is sketched in Appendix B.

The computations reported in Section 3 of the paper were carried out by Ralph Bailey to whom I am very grateful.

[2] Working with the same model, Ullah and Nagar [**22**] have derived an expression for the exact mean of the 2SLS estimator.

characterized the class $\bigcup_{n=0}^{\infty} H_{n,0}$ corresponding to a just identified equation containing $n+1$ endogenous variables; and in the same set up, in the overidentified case, Sargan represents the p.d.f. as an integral over a matrix space of dimension $n(n+1)/2$, which is not reduced.

The present paper is concerned with deriving the p.d.f. in the latter case and so characterizing in Basmann's notation, the class

$$H = \bigcup_{n=1}^{\infty} \bigcup_{N=0}^{\infty} H_{n,N}$$

corresponding to a structural equation containing any number of endogenous variables and an arbitrary number of degrees of overidentification. This generalizes all presently known results for single equation instrumental variable estimators in the simultaneous equations setting.

2. THE MODEL AND NOTATION

We will work with the structural equation

$$(1) \qquad y_1 = Y_2\beta + Z_1\gamma + u$$

where $y_1(T \times 1)$ and $Y_2(T \times n)$ are an observation vector and observation matrix, respectively, of $n+1$ included endogenous variables, Z_1 is a $T \times K_1$ matrix of included exogenous variables, and u is a random disturbance vector. The reduced form of (1) is given by

$$(2) \qquad [y_1 : Y_2] = [Z_1 : Z_2]\begin{bmatrix} \pi_{11} & \Pi_{12} \\ \pi_{21} & \Pi_{22} \end{bmatrix} + [v_1 : V_2] = Z\Pi + V,$$

where Z_2 is a $T \times K_2$ matrix of exogenous variables excluded from (1). The rows of the reduced form disturbance matrix V are assumed to be independent, identically distributed, normal random vectors. We assume that the usual standardizing transformations (Basmann [**2, 3**] and Richardson [**18**]) have been carried out so that the covariance matrix of rows of V is the identity matrix and $T^{-1}Z'Z = I_K$ where $K = K_1 + K_2$ and $Z = [Z_1 : Z_2]$. We also assume that $K_2 \geq n$ and the matrix $\Pi_{22}(K_2 \times n)$ in (2) has full rank, so that (1) is identified. We use the parameter $N = K_2 - n$ as in Sargan [**19**, Appendix B] to measure the degree of overidentification.

We let $H = [Z_1 : Z_3]$, where $Z_3(T \times K_3)$ is a submatrix of Z_2 and $K_3 \geq n$, be a matrix of instrumental variables to be used in the estimation of (1). We concentrate on the estimation of the parameter vector β and, from the orthogonality of the exogenous variables, we find that the estimator is the solution of

$$(Y_2'HH'Y_2 - Y_2'Z_1Z_1'Y_2)\beta_{IV} = Y_2'HH'y_1 - Y_2'Z_1Z_1'y_1$$

or

(3) $$(Y_2'Z_3Z_3'Y_2)\beta_{IV} = Y_2'Z_3Z_3'y_1.$$

3. THE EXACT DISTRIBUTION OF β_{IV}

We define

$$A = \begin{bmatrix} a_{11} & a_{21}' \\ a_{21} & A_{22} \end{bmatrix} = T^{-1}\begin{bmatrix} y_1'Z_3Z_3'y_1 & y_1'Z_3Z_3'Y_2 \\ Y_2'Z_3Z_3'y_1 & Y_2'Z_3Z_3'Y_2 \end{bmatrix}$$

and then $\beta_{IV} = A_{22}^{-1}a_{21}$. The starting point in our derivation of the joint p.d.f. of β_{IV} is to write down the joint distribution of the matrix A. This is noncentral Wishart of order $n+1$. Modern methods of multivariate analysis (based substantially on [**5**, **10**, and **13**]) enable us to employ a convenient mathematical representation of the joint density of the matrix A in terms of a matrix argument hypergeometric function (see (4) below). To obtain the p.d.f. of β_{IV} we then transform variates so that we are working directly with the function $A_{22}^{-1}a_{21}$. The final step in the derivation is to integrate over the space of (a_{11}, A_{22}) leaving us with the required density function for β_{IV}.

We write $A = T^{-1}X'Z_3Z_3'X$ where $X = [y_1 : Y_2]$. The K_3 columns of $T^{-\frac{1}{2}}X'Z_3$ are independently distributed normal vectors with covariance matrix I_{n+1} and, from (2), $E(T^{-\frac{1}{2}}Z_3'X) = T^{-\frac{1}{2}}Z_3'Z\Pi = M'$, say. Thus, following James [**13**, p. 484], we can write the distribution of A as

(4) $$\text{p.d.f.}(A) = \frac{\operatorname{etr}(-\frac{1}{2}MM')}{2^{\frac{1}{2}K_3(n+1)}\Gamma_{n+1}\left(\frac{K_3}{2}\right)}\,{}_0F_1\left(\frac{K_3}{2}; \tfrac{1}{4}MM'A\right) \cdot \operatorname{etr}(-\tfrac{1}{2}A)(\det A)^{\frac{1}{2}(K_3-n-2)}$$

where etr( ) denotes the operator exp(tr( )), $\Gamma_m(a)$ is the multivariate Gamma function,[3] and ${}_0F_1(\ ;\ )$ is a matrix argument hypergeometric function (closely related to the Bessel function of matrix argument discussed by Herz [**10**]). Constantine [**5**] discovered that hypergeometric functions ${}_pF_q$ of matrix argument have a series representation in terms of zonal polynomials (James [**12**, **13**]) as follows:

(5) $${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; S) = \sum_{j=0}^{\infty}\sum_J \frac{(a_1)_J \ldots (a_p)_J}{(b_1)_J \ldots (b_q)_J}\frac{C_J(S)}{j!}$$

[3] $$\Gamma_m(a) = \int_{S>0} \operatorname{etr}(-S)(\det S)^{a-\frac{1}{2}(m+1)}\,dS,$$

where the integration is over the set of all positive definite symmetric matrices; and in terms of the univariate Gamma function

$$\Gamma_m(a) = \pi^{\frac{1}{4}m(m-1)}\prod_{i=1}^{m}\Gamma(a - \tfrac{1}{2}(i-1)) \qquad \text{(James [13, p. 483])}.$$

where J indicates a partition of the integer j into not more than m parts, when S is an $m \times m$ matrix. If $J = \{j_1, j_2, \ldots, j_m\}$, then the coefficients appearing in (5) are given by

$$(a)_J = \prod_{i=1}^{m} (a - \tfrac{1}{2}(i-1))_{j_i}$$

where

$$(\lambda)_j = \lambda(\lambda+1)\ldots(\lambda+j-1).$$

The factor $C_J(S)$ in (5) is a zonal polynomial and can be represented as a symmetric, homogeneous polynomial of the latent roots of S of degree j. General formulae for these polynomials are known only for the case $m = 2$ but tabulations are currently available for values of j up to 12 (James [**14**]).[4]

We now introduce a matrix S which selects those columns of Z_2 which appear in Z_3, so that $Z_3 = Z_2 S$. Then, using the orthogonality of the exogenous variables, we have

$$M' = T^{-\frac{1}{2}} Z_3' Z\Pi = T^{\frac{1}{2}}[O\ S']\Pi = T^{\frac{1}{2}} S'[\pi_{21}\ \Pi_{22}] = T^{\frac{1}{2}} S' \Pi_{22}[\beta, I]$$

in view of the relationship between (1) and (2). Writing $\Pi_{22}' SS' \Pi_{22}$ as $\bar{\Pi}_{22}'\bar{\Pi}_{22}$ where $\bar{\Pi}_{22}$ is an $n \times n$ matrix (which is nonsingular since the structural equation (1) is identified), we find that

$$\operatorname{etr}(-\tfrac{1}{2}MM') = \operatorname{etr}\left\{-\frac{T}{2}(I + \beta\beta')\,\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}.$$

Moreover, since the nonzero latent roots of $MM'A$ are the latent roots of

$$T\bar{\Pi}_{22}[\beta, I]A\begin{bmatrix}\beta' \\ I\end{bmatrix}\bar{\Pi}_{22}',$$

(4) becomes

$$\frac{\operatorname{etr}\left\{-\dfrac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}}{2^{\frac{1}{2}K_3(n+1)}\Gamma_{n+1}\left(\dfrac{K_3}{2}\right)}$$

$$\cdot\, {}_0F_1\left(\frac{K_3}{2}; \frac{T}{4}\bar{\Pi}_{22}[\beta, I]A\begin{bmatrix}\beta' \\ I\end{bmatrix}\bar{\Pi}_{22}'\right) \cdot \operatorname{etr}(-\tfrac{1}{2}A)(\det A)^{\frac{1}{2}(K_3-n-2)}.$$

[4] For further discussion of zonal polynomials, see James [**12, 13**] and Johnson and Kotz [**15**]; and, for a recent survey of known results on zonal polynomials, Subrahmaniam [**21**] is a useful reference.

We now transform variables from the matrix variate A to $w = a_{11} - r'A_{22}r$, $r = A_{22}^{-1}a_{21}$, and $A_{22} = A_{22}$. The Jacobian of the transformation is det A_{22} and we have

$$\text{p.d.f.}(w, r, A_{22}) = \frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}}{2^{\frac{1}{2}K_3(n+1)}\Gamma_{n+1}\left(\frac{K_3}{2}\right)} \cdot {}_0F_1\left(\frac{K_3}{2}; \frac{T}{4}\left\{w\bar{\Pi}_{22}\beta\beta'\bar{\Pi}_{22}'\right.\right.$$
$$\left.\left.+\bar{\Pi}_{22}(I+\beta r')A_{22}(I+r\beta')\bar{\Pi}_{22}'\right\}\right)$$
$$\cdot \exp(-\tfrac{1}{2}(w + r'A_{22}r))\operatorname{etr}(-\tfrac{1}{2}A_{22})$$
$$\cdot \{w \det A_{22}\}^{\frac{1}{2}(K_3-n-2)}\det A_{22}$$
$$= \frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}}{2^{\frac{1}{2}K_3(n+1)}\Gamma_{n+1}\left(\frac{K_3}{2}\right)}$$
$$\cdot {}_0F_1\left(\frac{K_3}{2}; \left\{\frac{T}{4}w\bar{\Pi}_{22}\beta\beta'\bar{\Pi}_{22}'\right.\right.$$
$$\left.\left.+\bar{\Pi}_{22}(I+\beta r')A_{22}(I+r\beta')\bar{\Pi}_{22}'\right\}\right)$$
$$\cdot \exp(-\tfrac{1}{2}w)\operatorname{etr}\{-\tfrac{1}{2}(I+rr')A_{22}\}$$
$$\cdot w^{\frac{1}{2}(K_3-n-2}(\det A_{22})^{\frac{1}{2}(K_3-n)}.$$

Define $L = K_3 - n$ and introduce the new matrix variate $B = (I+rr')^{\frac{1}{2}}A_{22}(I+rr')^{\frac{1}{2}}$. The Jacobian of this transformation is $[\det(I+rr')]^{(n+1)/2}$ and we have

$$\text{(6)}\qquad \text{p.d.f.}(w, r, B) = \frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}}{2^{\frac{1}{2}(L+n)(n+1)}\Gamma_{n+1}\left(\frac{L+n}{2}\right)[\det(I+rr')]^{[L+n+1)/2}}$$
$$\cdot {}_0F_1\left(\frac{L+n}{2}; \frac{T}{4}\left\{w\bar{\Pi}_{22}\beta\beta'\bar{\Pi}_{22}' + \bar{\Pi}_{22}(I+\beta r')(I+rr')^{-\frac{1}{2}}\right.\right.$$
$$\left.\left.\cdot B(I+rr')^{-\frac{1}{2}}(I+r\beta')\bar{\Pi}_{22}'\right\}\right)$$
$$\cdot \exp(-\tfrac{1}{2}w)\operatorname{etr}(-\tfrac{1}{2}B)w^{L/2-1}(\det B)^{L/2}.$$

We now use the following inverse Laplace transform representation of the ${}_0F_1$ function (James [13, p. 480]):

$$\text{(7)}\qquad {}_0F_1(b; S) = \frac{2^{\frac{1}{2}n(n-1)}\Gamma_n(b)}{(2\pi i)^{\frac{1}{2}n(n+1)}}\int_{\operatorname{Re}(R)=X_0>0}\operatorname{etr}(R)(\det R)^{-b}\operatorname{etr}(R^{-1}S)\,dR$$

where the integral is taken over all matrices $R = X_0 + iY$ with X_0 a fixed positive definite matrix and Y ranging over all real symmetric matrices. Using this representation in (6) we have

$$\text{(8)} \qquad \text{p.d.f.}(w, r, B) = \frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}'_{22}\bar{\Pi}_{22}\right\}}{2^{\frac{1}{2}(L+n)(n+1)}\Gamma_{n+1}\left(\frac{L+n}{2}\right)[\det(I+rr')]^{(L+n+1)/2}}$$

$$\cdot \frac{2^{\frac{1}{2}n(n-1)}\Gamma_n\left(\frac{L+n}{2}\right)}{(2\pi i)^{\frac{1}{2}n(n+1)}} \int_{\operatorname{Re}(R)=X_0>0} \operatorname{etr}(R)(\det R)^{-(L+n)/2}$$

$$\cdot \operatorname{etr}\left\{\frac{Tw}{4}R^{-1}\bar{\Pi}_{22}\beta\beta'\bar{\Pi}'_{22}\right\}$$

$$\cdot \operatorname{etr}\left\{\frac{T}{4}R^{-1}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-\frac{1}{2}}\right.$$

$$\left.\cdot B(I+rr')^{-\frac{1}{2}}(I+r\beta')\bar{\Pi}'_{22}\right\} dR$$

$$\cdot \exp(-\tfrac{1}{2}w)\operatorname{etr}(-\tfrac{1}{2}B)w^{L/2-1}(\det B)^{L/2}.$$

We now integrate out $w (0 < w < \infty)$ and the matrix variate $B (B > 0)$ in (8). We have

$$\int_{B>0} \operatorname{etr}(-\tfrac{1}{2}B)(\det B)^{L/2} \operatorname{etr}\left\{\frac{T}{4}R^{-1}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-\frac{1}{2}}\right.$$

$$= 2^{L/2}\Gamma\left(\frac{L}{2}\right)\left(1-\frac{T}{2}\beta'\bar{\Pi}'_{22}R^{-1}\bar{\Pi}_{22}\beta\right)^{-L/2}$$

and

$$\int_{B>0} \operatorname{etr}(-\tfrac{1}{2}B)(\det B)^{L/2}\operatorname{etr}\left\{\frac{T}{4}R^{-1}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-\frac{1}{2}}\right.$$

$$\left.\cdot B(I+rr')^{-\frac{1}{2}}(I+r\beta')\bar{\Pi}'_{22}\right\} dR$$

$$= 2^{n(L+n+1)/2}\Gamma_n\left(\frac{L+n+1}{2}\right) {}_1F_0\left(\frac{L+n+1}{2}; \frac{T}{2}(I+rr')^{-\frac{1}{2}}\right.$$

$$\left.\cdot (I+r\beta')\bar{\Pi}'_{22}R^{-1}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-\frac{1}{2}}\right).$$

The latter integral follows, for example, by transforming $\frac{1}{2}B \to B$ and employing the Laplace transform given by James [**13**, p. 480, equation (28)].

Now we obtain

$$\text{(9)} \qquad \text{p.d.f.}(r) = \frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\} 2^{\frac{1}{2}L+\frac{1}{2}n(L+n+1)}\Gamma\left(\frac{L}{2}\right)\Gamma_n\left(\frac{L+n+1}{2}\right)}{2^{\frac{1}{2}(L+n)(n+1)}\Gamma_{n+1}\left(\frac{L+n}{2}\right)[\det(I+rr')]^{(L+n+1)/2}}$$

$$\cdot \frac{2^{\frac{1}{2}n(n-1)}\Gamma_n\left(\frac{L+n}{2}\right)}{(2\pi i)^{\frac{1}{2}n(n+1)}} \int_{\operatorname{Re}(R)=X_0>0} \operatorname{etr}(R)(\det R)^{-(L+n)/2}$$

$$\cdot \left(1-\frac{T}{2}\beta'\bar{\Pi}_{22}'R^{-1}\bar{\Pi}_{22}\beta\right)^{-L/2} {}_1F_0\left(\frac{L+n+1}{2}; \frac{T}{2}(I+rr')^{-\frac{1}{2}}\right.$$

$$\left.\cdot (I+r\beta')\bar{\Pi}_{22}'R^{-1}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-\frac{1}{2}}\right) dR.$$

Noting that

$$\Gamma_{n+1}\left(\frac{L+n}{2}\right) = \pi^{n/2}\Gamma_n\left(\frac{L+n}{2}\right)\Gamma\left(\frac{L}{2}\right)$$

and expanding the last two factors in the integrand in (9), we have the following integral representation of the density:

$$\text{(10)} \qquad \text{p.d.f.}(r) = \frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}\Gamma_n\left(\frac{L+n+1}{2}\right)}{\pi^{n/2}[\det(I+rr')]^{(L+n+1)/2}}$$

$$\cdot \sum_{j=0}^{\infty} \frac{\left(\frac{L}{2}\right)_j}{j!} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \left(\frac{L+n+1}{2}\right)_{\kappa} \frac{2^{\frac{1}{2}n(n-1)}}{(2\pi i)^{\frac{1}{2}n(n+1)}}$$

$$\cdot \int_{\operatorname{Re}(R)=X_0>0} \operatorname{etr}(R)(\det R)^{-(L+n)/2}\left(\frac{T}{2}\beta'\bar{\Pi}_{22}'R^{-1}\bar{\Pi}_{22}\beta\right)^j$$

$$\cdot C_{\kappa}\left(\frac{T}{2}(I+rr')^{-\frac{1}{2}}(I+r\beta')\bar{\Pi}_{22}'R^{-1}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-\frac{1}{2}}\right) dR.$$

To find the joint density of β_{IV} it remains to evaluate the inverse Laplace transform that occurs in the right-hand side of (10). The problem is related to the inverse Laplace transform of a zonal polynomial discussed by Constantine [**5**] but is more complicated in view of the presence of both a zonal polynomial and a power of a quadratic form involving R^{-1} in the integrand. However, this can be overcome by introducing a matrix of auxiliary variables as shown in the Appendix, where I give a procedure that enables us to evaluate the integral directly.

In particular, using (A5) in (10) by setting $t=(L+n)/2$, $g=(T/2)^{\frac{1}{2}}\bar{\Pi}_{22}\beta$, and $G=(T/2)^{\frac{1}{2}}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-\frac{1}{2}}$, we obtain

$$\text{(11)}\qquad \text{p.d.f.}(r)=\frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}\Gamma_n\left(\frac{L+n+1}{2}\right)}{\pi^{n/2}[\det(I+rr')]^{(L+n+1)/2}}$$

$$\cdot\sum_{j=0}^{\infty}\frac{\left(\frac{L}{2}\right)_j}{j!}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\kappa}\frac{\left(\frac{L+n+1}{2}\right)_\kappa}{\Gamma_n\left(\frac{L+n}{2}+j,\kappa\right)}$$

$$\cdot\left[\left(\frac{T}{2}\beta'\bar{\Pi}_{22}'\left(\operatorname{adj}\frac{\partial}{\partial W}\right)\bar{\Pi}_{22}\beta\right)^j(\det(I+W))^{(L+n)/2+j-(n+1)/2}\right.$$

$$\left.\cdot\, C_\kappa\left(\frac{T}{2}(I+W)\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}_{22}'\right)\right]_{W=0}$$

where the constant $\Gamma_n((L+1)/2)+j,\kappa)$ appearing in (11) is defined in (A1) of the Appendix. From Constantine [**5**] (equation (27)) we have the representation of the generalized binomial coefficients as $(a)_\kappa=\Gamma_n(a,\kappa)/\Gamma_n(a)$, which enables us to write (11) as

$$\text{(12)}\qquad \text{p.d.f.}(r)=\frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}\Gamma_n\left(\frac{L+n+1}{2}\right)}{\pi^{n/2}[\det(I+rr')]^{(L+n+1)/2}}$$

$$\cdot\sum_{j=0}^{\infty}\frac{\left(\frac{L}{2}\right)_j}{j!\,\Gamma_n\left(\frac{L+n}{2}+j\right)}\left[\left(\frac{T}{2}\beta'\bar{\Pi}_{22}'\left(\operatorname{adj}\frac{\partial}{\partial W}\right)\bar{\Pi}_{22}\beta\right)^j\right.$$

$$\cdot(\det(I+W))^{(L+n)/2+j-(n+1)/2}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\left(\frac{L+n+1}{2}\right)_\kappa}{\left(\frac{L+n}{2}+j\right)_\kappa}$$

$$\left.\cdot\, C_\kappa\left(\frac{T}{2}(I+W)\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}_{22}'\right)\right]_{W=0}$$

$$=\frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}\Gamma_n\left(\frac{L+n+1}{2}\right)}{\pi^{n/2}[\det(I+rr')]^{(L+n+1)/2}}$$

$$\cdot \sum_{j=0}^{\infty} \frac{\left(\frac{L}{2}\right)_j}{j!\Gamma_n\left(\frac{L+n}{2}+j\right)} \left[\left(\frac{T}{2} \beta' \bar{\Pi}'_{22} \left(\operatorname{adj} \frac{\partial}{\partial W} \right) \bar{\Pi}_{22} \beta \right)^j \right.$$

$$\cdot (\det(I+W))^{(L-1)/2+j} {}_1F_1\left(\frac{L+n+1}{2}, \frac{L+n}{2} \right.$$

$$\left. \left. +j; \frac{T}{2}(I+W)\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}'_{22} \right) \right]_{W=0}.$$

If we set $L = N$ so that $Z_3 = Z_2$, β_{IV} is the 2SLS estimator and the joint density (14) generalizes the formula derived by Richardson [**18**] for the case where $n = 1$. In this case, the matrix W appearing in (14) is a scalar, $\operatorname{adj}(\partial/\partial W) = 1$, and we obtain

$$\text{(13)} \qquad \text{p.d.f.}(r) = \frac{\exp\left\{ -\frac{\mu^2}{2}(1+\beta^2) \right\} \Gamma\left(\frac{N+2}{2} \right)}{\pi^{\frac{1}{2}} \Gamma\left(\frac{N+1}{2} \right) (1+r^2)^{(N+2)/2}}$$

$$\cdot \sum_{j=0}^{\infty} \frac{\left(\frac{N}{2}\right)_j}{j! \left(\frac{N+1}{2} \right)_j} \left(\frac{\mu^2}{2} \beta^2 \right)^j {}_1F_1\left(\frac{N+2}{2}, \frac{N+1}{2}+j; \frac{\mu^2}{2} \frac{(1+\beta r)^2}{1+r^2} \right)$$

where $\mu^2 = T\bar{\Pi}_{22}^2 = T\Pi'_{22}\Pi_{22'}$, the concentration parameter. Noting that $N = K_2 - 1$ where K_2 is the number of exogenous variables excluded from (1), we see that (13) corresponds with the known formula in this case [**11, 18**].

When $L = 0$ in (12) the series corresponding to the suffix j terminates at the first term and we have

$$\text{(14)} \qquad \text{p.d.f.}(r) = \frac{\operatorname{etr}\left\{ -\frac{T}{2}(I+\beta\beta')\bar{\Pi}'_{22}\bar{\Pi}_{22} \right\} \Gamma_n\left(\frac{n+1}{2} \right)}{\pi^{n/2} \Gamma_n\left(\frac{n}{2} \right) [\det(I+rr')]^{(n+1)/2}}$$

$$\cdot {}_1F_1\left(\frac{n+1}{2}, \frac{n}{2}, \frac{T}{2} \bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}'_{22} \right),$$

that is, a single term involving a matrix argument hypergeometric function as obtained by Sargan [**19**] in this special case.

3. FURTHER COMMENTS AND SOME COMPUTATIONS FOR THE CASE $n = 2$

While (12) gives us a general representation of the exact joint density function of instrumental variable estimators in simultaneous equation models, this type of series representation of the density is not as easy to interpret as we would like. It can be said that the leading term in the density reveals the order to which finite sample moments of the estimator exist (c.f. Basmann [**3**]). In the present case, we see that when $L=0$ the leading term involves $[\det(I+rr')]^{-(n+1)/2} = (1+r'r)^{-(n+1)/2}$, which is proportional to the multivariate Cauchy density [**15**]; when $L>0$ the term involves $[\det(I+rr')]^{-[L+n+1)/2} = (1+r'r)^{-(L+n+1)/2}$ which is similar to a multivariate t density [**15**]. These expressions enable us to verify directly Basmann's conjecture [**1, 2**] that integer moments of the 2SLS estimator ($L=N$) exist up to the degree of overidentification.[5] In other respects, the analytic form of (12) is not very revealing. A similar comment applies to the alternative representation of the density (B3) given in Appendix B, which gives an explicit series representation of the density in terms of polynomials of two matrix arguments (which we term the Davis polynomials). Moreover series representations such as (12), (14), and (B3) cannot be implemented for numerical calculations as easily as might be expected. The formulae rely on matrix argument functions and numerical evaluation depends on the available tabulations of zonal polynomials and the Davis polynomials. In most cases these will be insufficient to secure reliable numerical values for the density.[6,7]

However, when $n=2$ an explicit representation of the matrix argument ${}_1F_1$ function (in terms of the univariate ${}_1F_1$ function and the elementary symmetric functions of the matrix) has been given by Muirhead [**17**]. This should facilitate computations in the $n=2$ case. An alternative explored by Sargan [**19**] in the case $n \geq 1$ and by Holly and Phillips [**11**] in the case $n=1$ is to simplify the formulae for the joint density by replacing the ${}_1F_1$ function with an approximation based on the first few terms of its asymptotic expansion.[8] The computations in Holly and Phillips [**11**] indicate that this type of approximation attains a high level of accuracy[9] against the exact density, in both the tail areas as well as the body of the distribution and for a wide range of parameter values. It may, therefore, be

[5] This conjecture has, of course, been verified earlier by Mariano [**16**] for even order moments and Hatanaka [**9**] in the case of both odd and even moments. Basmann [**3**] recently discussed the role of the leading term in these series representations of density functions in determining the order to which moments exist.

[6] Zonal polynomials are currently tabulated up to order 12 and the Davis polynomials to order 6. Experience with these series in the case $n=1$ suggests that it may often be necessary to include as many as 100 terms before achieving adequate convergence. Muirhead [**17**] has already made a similar comment.

[7] In the case of (12) we have the additional complication of applying the matrix differential operator $\operatorname{adj}(\partial/\partial W)$ to the zonal polynomials appearing in the series representation of the ${}_1F_1$ function. When $n=2$ the zonal polynomial $C_\kappa(S)$ can be written in terms of the two elementary symmetric functions $\operatorname{tr}(S)$, $\det(S)$ of the matrix S (James [**13**, equation (130)]). In this case, the operation should not be too difficult. But when $n>2$, the complexity of this operation will certainly increase.

[8] Muirhead and Constantine [**6**] give the general form of this expansion for the matrix argument ${}_1F_1$ function as well as other hypergeometric functions.

[9] The computations in [**11**] show that 3 decimal place accuracy can be obtained with this approximation over a wide region of the distribution including tail areas.

worthwhile to pursue this alternative in the present case for general n and illustrate some numerical computations in the special case $n=2$.

From Muirhead and Constantine [6, Theorem 3.2, pp. 377] we have the approximation

$$ {}_1F_1(a, b;\, TR) \sim \frac{\Gamma_n(b)}{\Gamma_n(a)} \operatorname{etr}(TR)(\det TR)^{a-b}[1+0(T^{-1})]. $$

We set

$$ D(r) = \bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}'_{22} $$

and then we have

$$ \text{p.d.f.}(r) \sim \frac{\operatorname{etr}\left\{-\dfrac{T}{2}(I+\beta\beta')\bar{\Pi}'_{22}\bar{\Pi}_{22}\right\}}{\pi^{n/2}[\det(I+rr')]^{(L+n+1)/2}} $$

$$ \cdot \sum_{j=0}^{\infty} \frac{\left(\frac{L}{2}\right)_j}{j!} \operatorname{etr}\left\{\frac{T}{2}D(r)\right\}\left(\det\left(\frac{T}{2}D(r)\right)\right)^{\frac{1}{2}-j} $$

$$ \cdot \left[\left(\frac{T}{2}\beta'\bar{\Pi}'_{22}\left(\operatorname{adj}\frac{\partial}{\partial W}\right)\bar{\Pi}_{22}\beta\right)^j \right. $$

$$ \left. \cdot (\det(I+W))^{L/2} \operatorname{etr}\left\{\frac{T}{2}WD(r)\right\}\right]_{W=0} $$

$$ \sim \frac{\operatorname{etr}\left\{-\dfrac{T}{2}(I+\beta\beta')\bar{\Pi}'_{22}\bar{\Pi}_{22}\right\}\operatorname{etr}\left\{\dfrac{T}{2}D(r)\right\}\left(\det\left(\dfrac{T}{2}D(r)\right)\right)^{\frac{1}{2}}}{\pi^{n/2}[\det(I+rr')]^{(L+n+1)/2}} $$

$$ \cdot \sum_{j=0}^{\infty} \frac{\left(\frac{L}{2}\right)_j}{j!} (\beta'\bar{\Pi}'_{22}D(r)^{-1}\bar{\Pi}_{22}\beta)^j $$

which, after a little simplification, is

$$ \frac{T^{n/2}\operatorname{etr}\left\{-\dfrac{T}{2}\dfrac{\bar{M}(r-\beta)(r-\beta)'}{1+r'r}\right\}(\det \bar{M})^{\frac{1}{2}}(1+r'\beta)}{2^{n/2}\pi^{n/2}(1+r'r)^{(L+n+2)/2}} $$

$$ \cdot \sum_{j=0}^{\infty} \frac{\left(\frac{L}{2}\right)_j}{j!} (\beta'(I+r\beta')^{-1}(I+rr')(I+\beta r')^{-1}\beta)^j $$

where $\bar{M} = \bar{\Pi}'_{22}\bar{\Pi}_{22}$. The series converges and the approximation is valid with a relative error of $0(T^{-1})$ provided

$$\beta'(I+r\beta')^{-1}(I+rr')(I+\beta r')^{-1}\beta = \frac{\beta'\beta+(r'\beta)^2}{(1+r'\beta)^2}<1,$$

that is, provided

$$1+2\beta'r-\beta'\beta>0.$$

For values of r satisfying this inequality we then have the approximation

$$\text{(15)} \qquad \text{p.d.f.}(r) \sim \frac{T^{n/2}\operatorname{etr}\left\{-\frac{T}{2}\frac{\bar{M}(r-\beta)(r-\beta)'}{1+r'r}\right\}(\det \bar{M})^{\frac{1}{2}}}{2^{n/2}\pi^{n/2}(1+r'r)^{(L+n+2)/2}} \frac{(1+\beta'r)^{L+1}}{(1+2\beta'r-\beta'\beta)^{L/2}}.$$

This formula corresponds with the dominant term of formula B14 of Sargan [**19**] (in a somewhat different notation) and with formula (25) of Holly and Phillips [**11**] (for the 2SLS case and where $n=1$).

The approximate joint density (15) can be used quite readily for numerical computations. Moreover, when $n=2$ marginal densities can also be computed after a one-dimensional numerical integration. Some graphs which illustrate the effect of changes in the parameters on the marginal density of the first component of β_{IV} (viz. β_{1IV} or r_1) when $n=2$ are given in Figures 1–4. Figures 5 and 6 provide graphs of the density in the case of only two endogenous variables (i.e. $n=1$) and therefore help to illustrate the effect on the distribution of the estimator of the inclusion of another endogenous variable in the equation.

Some of the features that emerge from the densities graphed in Figures 1 to 6 are as follows:

(i) For comparable parameter values such as those in Figures 1 and 5 we note that the distribution of the estimator is more concentrated in the two endogenous variable case ($n=1$). The distribution also appears to concentrate more quickly as T becomes large when $n=1$ than when $n=2$.

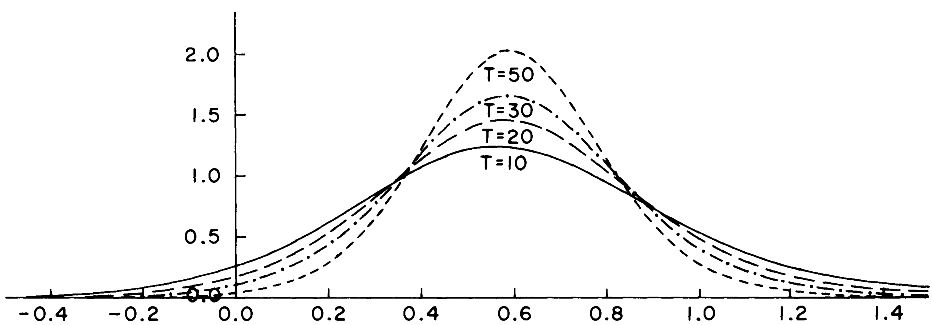

FIGURE 1.—Densities of β_{1IV} for various values of T when

$$n=2, \qquad \begin{matrix}\beta_1=0.6,\\ \beta_2=0.6,\end{matrix} \qquad \bar{M}=\begin{bmatrix}4.0 & 3.6\\ 3.6 & 4.0\end{bmatrix}, \qquad L=3.$$

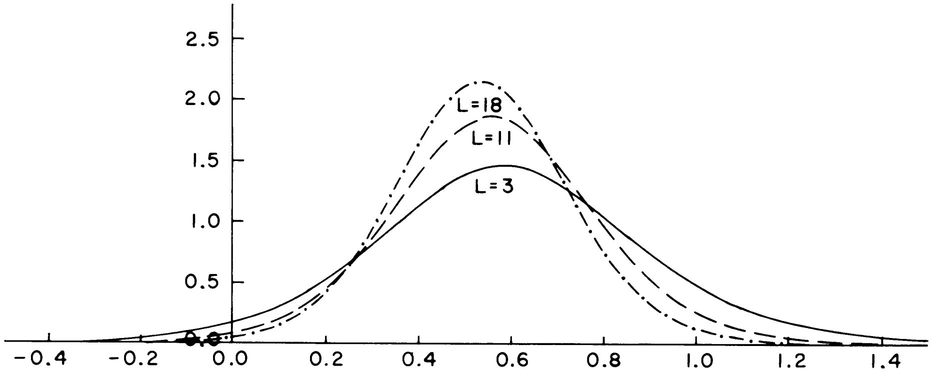

FIGURE 2.—Densities of β_{1IV} for various values of L when

$$n=2, \quad \begin{matrix}\beta_1=0.6,\\ \beta_2=0.6,\end{matrix} \quad \bar{M}=\begin{bmatrix}4.0 & 3.6\\ 3.6 & 4.0\end{bmatrix}, \quad T=20.$$

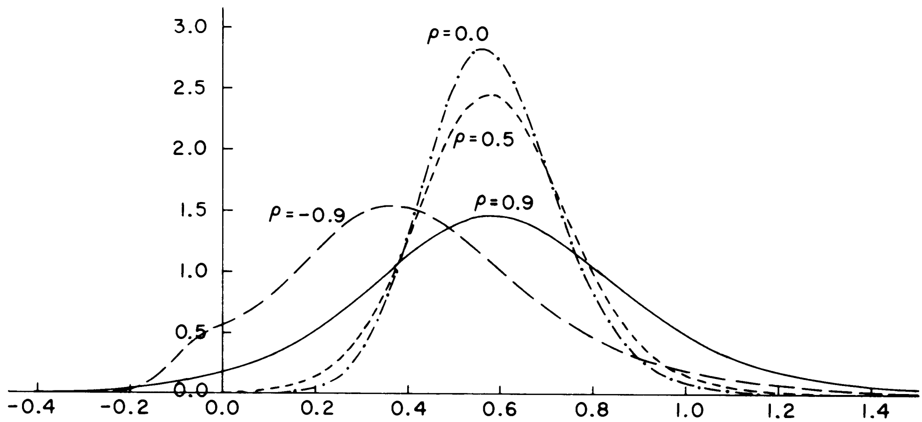

FIGURE 3.—Densities of β_{1IV} for various degrees of correlation ρ in $\bar{M}$ when

$$n=2, \quad \begin{matrix}\beta_1=0.6,\\ \beta_2=0.6,\end{matrix} \quad \bar{M}=\begin{bmatrix}4.0 & \rho 4.0\\ \rho 4.0 & 4.0\end{bmatrix}, \quad T=20, \quad L=3.$$

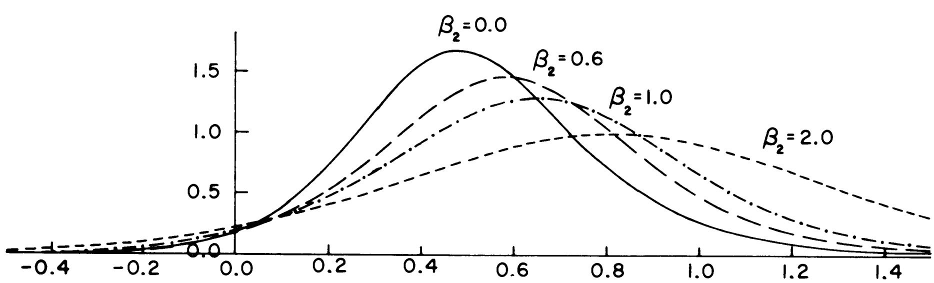

FIGURE 4.—Densities of β_{1IV} for various values of β_2 when

$$n=2, \quad \beta_1=0.6, \quad \bar{M}=\begin{bmatrix}4.0 & 3.6\\ 3.6 & 4.0\end{bmatrix}, \quad T=20, \quad L=3.$$

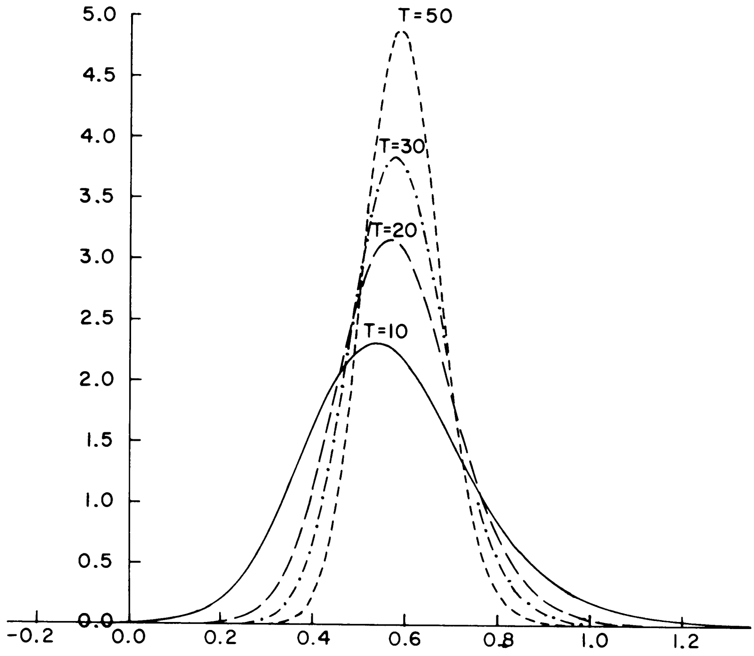

FIGURE 5.—Densities of β_{IV} for various values of T when $n=1$, $\beta_1=0.6$, $\bar{M}=4$, $L=3$.

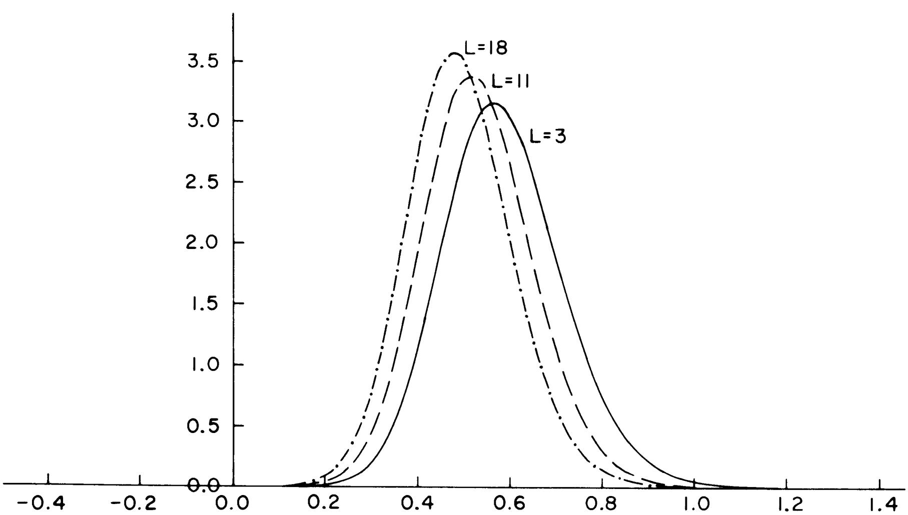

FIGURE 6.—Densities of β_{IV} for various values at L when $n=1$, $\beta_1=0.6$, $\bar{M}=4.0$, $T=20$.

(ii) From Figure 2 we see that the density of β_{1IV} displays more bias as L, the number of additional instruments used for the n right hand side endogenous variables, increases in value. When L becomes small the distribution is more centrally located about the true value of the parameter but also has greater dispersion than when L is large. These properties seem to correspond fairly well with the distribution in the two endogenous variable case (Fig. 6).

(iii) The density appears to be sensitive to the degree of correlation, ρ, in the matrix of products of reduced form coefficients $\bar{M} = \bar{\Pi}'_{22}\bar{\Pi}_{22}$. The dispersion of the density seems to increase with $|\rho|$. The density is well centered about the true value of the parameter when $\rho > 0$ but biased downwards when $\rho > 0$.

(iv) When $\beta_2 \neq \beta_1$ the distribution of β_{1IV} becomes less well centered about the true value of β_1. As $\beta_2 - \beta_1$ increases in value the bias becomes positive and the dispersion increases rapidly; as $\beta_2 - \beta_1$ decreases the bias becomes negative and the distribution becomes more concentrated.

Yale University

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APPENDIX A

We start with the following result due to Constantine [**5**, p. 1273]:

LEMMA: *If R is a complex symmetric matrix whose real part is positive definite and Q is an arbitrary complex symmetric matrix, then*

$$\text{(A1)}\qquad \int_{S>0} etr(-RS)(\det S)^{t-(n+1)/2}C_\kappa(SQ)\,dS$$
$$= \Gamma_n(t,\kappa)(\det R)^{-t}C_\kappa(QR^{-1});\ \Gamma_n(t,\kappa) = \pi^{\frac{1}{4}n(n-1)}\prod_{i=1}^{n}\Gamma(t+k_i-\tfrac{1}{2}(i-1))$$

where the integration is over the space of positive definite $n \times n$ matrices and is valid for all complex t for which $Re(t) > (n-1)/2$.

If we write

$$g(R) = \Gamma_n(t,\kappa)(\det R)^{-t}C_\kappa(QR^{-1})$$

and

$$f(S) = (\det S)^{t-(n+1)/2}C_\kappa(SQ),$$

then according to (A1), $g(R)$ is the Laplace transform of $f(S)$.

We will be dealing with a case in which Q is positive definite. The inverse transform corresponding to (A1) is then given by the relation[10]

$$\text{(A2)}\qquad \frac{2^{\frac{1}{2}n(n-1)}}{(2\pi i)^{\frac{1}{2}n(n+1)}}\int_{\mathrm{Re}(R)>X_0} \mathrm{etr}(SR)(\det R)^{-t}C_\kappa(QR^{-1})\,dR$$
$$= [\Gamma_n(t,\kappa)]^{-1}(\det S)^{t-(n+1)/2}C_\kappa(SQ)$$

where X_0 is a fixed positive definite matrix and, writing $R = X + iY$ with $X > X_0$, the integration in (A2) is taken over all real symmetric matrices Y.

[10] (A2) can be verified directly from formula (21), p. 1275 of Constantine [**5**] by replacing S in Constantine's formula with $Q^{\frac{1}{2}}SQ^{\frac{1}{2}}$.

The integral we need to evaluate in the paper (in (10)) is not directly of the form (A2). Instead, we must consider integrals of the form

$$\text{(A3)}\qquad \int_{\mathrm{Re}(R)>X_0} \operatorname{etr}(R)(\det R)^{-t}(g'R^{-1}g)^j C_\kappa(G'R^{-1}G)\,dR$$

where j is an integer, g is a fixed n-vector and G is a nonsingular matrix of order n. To evaluate (A3) we note that

$$g'\left(\operatorname{adj}\frac{\partial}{\partial W}\right)g\ \operatorname{etr}(RW) = \operatorname{etr}(RW)(g'R^{-1}g)(\det R)$$

where W is an $n\times n$ matrix and $\operatorname{adj}(\partial/\partial W)$ denotes the adjoint of the matrix operator $\partial/\partial W$. Moreover, by repeated use of the operator $g'\operatorname{adj}(\partial/\partial W)g$ in the above equation, we obtain

$$\left\{g'\left(\operatorname{adj}\frac{\partial}{\partial W}\right)g\right\}^j \operatorname{etr}(RW) = \operatorname{etr}(RW)(g'R^{-1}g)^j(\det R)^j$$

and, thus,

$$\text{(A4)}\qquad \int_{\mathrm{Re}(R)>X_0} \operatorname{etr}(R)\,(\det R)^{-t}(g'R^{-1}g)^j C_\kappa(G'R^{-1}G)\,dR$$

$$= \left[\left\{g'\left(\operatorname{adj}\frac{\partial}{\partial W}\right)g\right\}^j \int_{\mathrm{Re}(R)>X_0} \operatorname{etr}((I+W)R)\,(\det R)^{-t-j} C_\kappa(G'R^{-1}G)\,dR\right]_{W=0}$$

Now, using (A2), we deduce the following relationship:

$$\text{(A5)}\qquad \frac{2^{\frac{1}{2}n(n-1)}}{(2\pi i)^{\frac{1}{2}n(n+1)}}\int_{\mathrm{Re}(n)>X_0} \operatorname{etr}(R)\,(\det R)^{-t}(g'R^{-1}g)^j C_\kappa(G'R^{-1}G)\,dR$$

$$= \left[\left\{g'\left(\operatorname{adj}\frac{\partial}{\partial W}\right)g\right\}^j \left[\Gamma_n(t+j,\kappa)\right]^{-1} (\det(I+W))^{t+j-(n+1)/2} C_\kappa((I+W)GG')\right]_{W=0}$$

$$= \frac{1}{\Gamma_n(t+j,\kappa)}\left[\left\{g'\left(\operatorname{adj}\frac{\partial}{\partial W}\right)g\right\}^j (\det(I+W))^{t+j-(n+1)/2}.\ C_\kappa((I+W)GG')\right]_{W=0}.$$

APPENDIX B

One advantage of the representation (12) obtained in the paper for the p.d.f. of β_{IV} is that this form of the density in the general case closely parallels that which is already known for the case $n=1$ (see (13)) and the latter can be simply deduced from it as a special case. However, the matrix differential operator that occurs in (12) complicates the series representation of the density and will make numerical computations of the exact density difficult even when extensive tabulations of zonal polynomials become available. In fact, an explicit series representation is possible by making use of the invariant polynomials of two matrix arguments that have recently been introduced by Davis [**7**, **8**]. This work has very kindly been brought to my attention by one of the referees. Since the derivations are brief it seems worthwhile to give the alternative representation of the density here.

We define $g=(T/2)^{\frac{1}{2}}\bar{\Pi}_{22}\beta$ and $G(r)=(T/2)^{\frac{1}{2}}\bar{\Pi}_{22}(I+\beta r')\,(I+rr')^{-\frac{1}{2}}$ as in the paper. We can now write the product $(g'R^{-1}g)^jC_\kappa(G(r)R^{-1}G(r)')$ that occurs in the integrand of (10) as a linear combination of the invariant polynomials introduced by Davis. We have ((5.8) in Davis [**7**])

$$\text{(B1)}\qquad (g'R^{-1}g)^jC_\kappa(G(r)R^{-1}G(r)')$$

$$= \sum_{\phi\in j,\kappa} \theta_\phi^{j,\kappa} C_\phi^{j,\kappa}(R^{-1}gg',\ R^{-1}G(r)'G(r))$$

where $C_\phi^{j,\kappa}(X,Y)$ is a polynomial in the elements of the two matrices X and Y which is invariant under the simultaneous transformation $X\to H'XH$ and $Y\to H'YH$ for any orthogonal matrix H. The partitions ϕ over which the summation in (B1) is taken is such that the representation of $Gl(n)$ (the general linear group of real nonsingular matrices of order n) indexed by 2ϕ occurs in the Kronecker product $2j \otimes 2\kappa$ of the representations indexed by $2j$ and 2κ. The constants $\theta_{\phi}^{j,\kappa}$ that occur in (B1) are given by Davis [7, equation (5.1)]:

$$\theta_{\phi}^{j,\kappa} = C_{\phi}^{j,\kappa}(I, I)/C_{\phi}(I).$$

From Davis [8, p. 4 we can deduce the following inverse Laplace transform:

$$\text{(B2)} \qquad \frac{2^{\frac{1}{2}n(n-1)}}{(2\pi i)^{\frac{1}{2}n(n+1)}} \int_{Re(R)=X_0>0} \operatorname{etr}(R)(\det R)^{-t} C_{\phi}^{j,\kappa}(R^{-1}gg', R^{-1}G(r)'G(r))\, dR$$
$$= [\Gamma_n(t, \phi)]^{-1} C_{\phi}^{j,\kappa}(gg', G(r)'G(r)).$$

Working from (10), (B1), and (B2) we obtain the alternative representation of the density in the paper as

$$\text{(B3)} \qquad \text{p.d.f.}(r) = \frac{\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}\Gamma_n\left(\frac{L+n+1}{2}\right)}{\pi^{n/2}[\det(I+rr')]^{(L+n+1)/2}}$$
$$\cdot \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\phi \in j,\kappa} \frac{\left(\frac{L}{2}\right)_j \left(\frac{L+n+1}{2}\right)_\kappa \theta_{\phi}^{j,\kappa}}{j!k!\Gamma_n\left(\frac{L+n}{2}, \phi\right)}$$
$$\cdot C_{\phi}^{j,\kappa}\left(\frac{T}{2}\bar{\Pi}_{22}\beta\beta'\bar{\Pi}_{22}', \frac{T}{2}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}_{22}'\right).$$

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