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THE CONCENTRATION ELLIPSOID OF A RANDOM VECTOR

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Alternative definitions of the concentration ellipsoid of a random vector are discussed, one of which helps to simplify the development of the theory of linear estimation.

1. Introduction

In the theory of linear estimation it is often useful to give a geometric representation of the dispersion of a random vector. Malinvaud's (1970, ch. 5) general treatment of the subject relies on this approach and makes extensive use of the concept of the concentration ellipsoid. The purpose of this note is to give an alternative representation of a concentration ellipsoid which simplifies the presentation of much of the theory.

2. A lemma

Suppose x is an n-dimensional random vector with mean vector y and covariance matrix Q. We have the following two definitions:

Definition 1. The concentration ellipsoid of x is the set

$$E=\{\varepsilon:|u'\varepsilon|\leqq 1 \text{ for all } u \text{ such that } u'Qu\leqq 1\}.$$

Definition 2. The concentration ellipsoid of x is the set

$$\tilde{E}=\{\varepsilon=Qv:v'Qv\leqq 1\}.$$

Definition 1 is used in Malinvaud (1970). The following lemma shows that Definitions 1 and 2 are equivalent:

Lemma. *Alternative representation of a concentration ellipsoid:*

$$\tilde{E}=E.$$

Proof. If $\varepsilon\in\tilde{E}$ then $\varepsilon=Qv$ for some v for which $v'Qv\leqq 1$. Now take $u\in\boldsymbol{R}^n$ for which $u'Qu\leqq 1$, and then

$$|u'\varepsilon|=|u'Qv|\leqq(u'Qu\,v'Qv)^{\frac{1}{2}}\leqq(v'Qv)^{\frac{1}{2}}\leqq 1.$$

Hence, $\varepsilon\in E$.

Conversely, if $\varepsilon\in E$ then $\varepsilon=Qv$ for some $v\in\boldsymbol{R}^n$ since E lies in the support, which is the range space of Q [Malinvaud (1970, p. 155)]. Now select $u\in\boldsymbol{R}^n$ for which $u'Qu=1$ and for which $Q^{\frac{1}{2}}u$ is collinear with $Q^{\frac{1}{2}}v$. Then

$$1\geqq|u'\varepsilon|=|u'Qv|=\|uQ^{\frac{1}{2}}\|\,\|Q^{\frac{1}{2}}v\|=(v'Qv)^{\frac{1}{2}},$$

so that $\varepsilon\in\tilde{E}$. Q.E.D.

3. Use of the lemma

The new definition is most useful in cases where the covariance matrix of the random vector may be singular. For instance, if

$$Q=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\\ \sigma_1\sigma_2 & \sigma_2^2\end{bmatrix},$$

then the support of $x-y$ (or range space of Q) is spanned by the vector $(\sigma_1,\sigma_2)'$. The concentration ellipsoid of x which lies within this support is just

$$\begin{aligned}E&=\{\varepsilon=Qv:v'Qv\leqq 1\}\\&=\left(\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}=\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix}(\sigma_1v_1+\sigma_2v_2):(\sigma_1v_1+\sigma_2v_2)^2\leqq 1\right)\\&=\left(\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}=\lambda\begin{bmatrix}\sigma_2\\ \sigma_2\end{bmatrix}:-1\leqq\lambda\leqq 1\right).\end{aligned}$$

In the general case where Q is singular and of rank $r<n$, we can use the definition [Malinvaud (1970, p. 156)]

$$E=\{\varepsilon:R'\varepsilon=0,\ \varepsilon'(Q+RR')^{-1}\varepsilon\leqq 1\},$$

where R is an $n \times (n-r)$ matrix of rank $n-r$ for which $QR=0$. Since $(Q+RR')^{-1}$ is a generalised inverse of Q and since $R'\varepsilon=0$ if and only if $\varepsilon=Qv$ for some vector v this alternative definition follows directly from Definition 2.

Definition 2 also enables us to develop alternative proofs of the main results in the theory of linear estimation. One fundamental result given in Malinvaud (1970, pp. 158–160) is the following: *If $f(x)=Gx$ is a linear transformation from one Euclidean space into another (G is the matrix of the transformation) then the concentration ellipsoid, E^*, of Gx is the image, under G, of the concentration ellipsoid, E, of x.*

Using Definition 1 it is easy to show that the image of E is contained in E^*. It is not so easy, using this definition, to verify that E^* is contained in the image of E. In fact, the proof of this property by Malinvaud (pp. 159–160) takes more than one page. However, consider the following proof of the latter property: *E^* is contained in the image of E.*

Suppose $\eta \in E^*$. Then $\eta = GQG'v$ for some v for which $v'GQG'v \leqq 1$. Now let $u=G'v$ and we can write $\eta=GQu$ with $u'Qu \leqq 1$ so that η lies in the image of E under G.

This example serves to illustrate how Definition 2 can be used to streamline some of the proofs of central results in the theory of linear estimation. Definition 2 also has the advantage of simplifying the algebraic representation of a concentration ellipsoid and corresponds more closely to the alternative definitions involving the inverse of Q when Q is non-singular and the generalised inverse of Q when Q is singular as shown above. It should therefore be most helpful in making more accessible the development of the theory of linear estimation along geometric lines.

Reference

Malinvaud, E., 1970, Statistical methods of econometrics (North-Holland, Amsterdam).