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A SADDLEPOINT APPROXIMATION TO THE DISTRIBUTION OF THE k-CLASS ESTIMATOR OF A COEFFICIENT IN A SIMULTANEOUS SYSTEM[1]

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A new approximation based on the saddlepoint method of approximating integrals is derived for the probability density of the k-class estimator in the case of the equation with two endogenous variables. The two tails of the density are approximated by different functions, each of which bears a close relationship with the exact density in the same region of the distribution. Corresponding approximations are also derived for the distribution function and the method of derivation should be useful in other applications. Some brief numerical results are reported which illustrate the performance of the new approximation.

1. INTRODUCTION

SEVERAL AUTHORS HAVE RECENTLY OBTAINED approximations to the distributions of coefficients of a single equation in a simultaneous system. The case of an equation with two endogenous variables has been intensively studied by Anderson and Sawa [**3**, **4**], Anderson [**1**, **2**], and Mariano [**18**, **19**]. The approximations used in these studies have been based on the first few terms of Edgeworth type asymptotic expansions of the distribution function of the estimator under consideration. This type of approximation has the appealing property that, if the series expansion from which it is derived is valid (i.e., the series is a proper asymptotic series in the mathematical sense), then the error on the approximation tends to zero as a key parameter (usually the sample size or the concentration parameter[2]) tends to infinity at a faster rate then the corresponding error on the asymptotic normal distribution.

One difficulty that has been experienced with approximations based on the Edgeworth expansion is that they can sometimes be unsatisfactory in the tail area.[3] In this region the errors on the approximation can be as large as the density ordinates or the tail probabilities themselves and the approximating density can well take on negative values. An alternative approach explored by Daniels [**7**, **8**] is the use of the saddlepoint approximation, which is always positive and has the same accuarcy (in terms of the order of magnitude of the error on the approximation) as the first two terms of the Edgeworth expansion. Moreover, the saddlepoint approximation is itself the first term in an asymptotic series expansion

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[2] Anderson [**2**] relates the expansions obtained for a number of different parameter sequences in the case of the LIML and 2SLS estimates.

[3] The results on Phillips [**20**, **21**] provide some recent evidence of this.

of the true probability density and can, therefore, be modified to take account of higher order terms. An important difference between the asymptotic series for which the saddlepoint approximation is the first term and the Edgeworth series from which the Edgeworth approximation is derived is that successive terms of the former have a smaller order of magnitude as the key parameter tends to infinity than corresponding terms of the latter. Thus, in the saddlepoint expansion (2) below we see that the first term is the saddlepoint approximation itself and the second term in the series is of $O(\mu^{-2})$ relative to the first where μ^2 is the concentration parameter (defined in (8)); whereas in the Edgeworth expansion the first term is the asymptotic normal approximation and the second term is of $O(\mu^{-1})$ (see, for instance, [**3**, equation (8.3)]). These features make the saddlepoint approximation an interesting alternative when it is available.

Since the saddlepoint approximation has not been used in previous econometric work[4] we provide some discussion of the principles underlying the method in Section 3 of the paper.[5] In the same section we derive the new approximation to the probability density of the k class estimator (k nonstochastic) in the case of an equation with two endogenous variables. Our approximation in the general case is based on the hypothesis that μ^2 becomes large when the sample size T is fixed,[6] but, in the special case of the two stage least squares (2SLS) estimator, it is valid also in situations where the sample size may increase with the concentration parameter.

The new approximation has the useful features just mentioned of being everywhere nonnegative and possessing an error of order μ^{-2} (equal to that of the first two terms of the Edgeworth expansion). Moreover, the order of magnitude of the error on the saddlepoint approximation holds in the *relative* sense, so that if $f(r)$ is the true density and $h(r)$ is the approximation we have $f(r)=h(r)[1+O(\mu^{-2})]$ as in (20) from which we can deduce that $(f(r)-h(r))/f(r)=O(\mu^{-2})$. This makes the saddlepoint approximation an attractive candidate for approximating the tail of the distribution, where $f(r)$ is small.

One new feature of the approximation in the present case that may be of interest is that the right hand and left hand tails of the exact density are approximated by different functions. In technical terms, this results from the fact that the saddlepoint we select (when approximating the integral representing the exact density) itself depends on the region of the density we are considering. In practice, this latter feature means that we can establish a very simple relationship between the exact density and our approximation.

For, the exact density can be written as a doubly infinite series in the case of the OLS and 2SLS estimators. This series can be written as a single infinite series of confluent hypergeometric functions in two different ways; and, as we show in Section 4 of the paper, the series can then be summed if we take the first term (or

[4] However, some recent work on saddlepoint approximations in simple time series models has been done by Durbin [**11**] and Phillips [**21, 24**].

[5] And in Appendix A also.

[6] Mariano also considers this case and discusses various situations in which this type of sequence is relevant [**19**, p., 720, footnote 6].

first few terms) of an asymptotic expansion of the confluent hypergeometric function. However, in order to obtain a convergent series we must select the appropriate representation of the double series and this choice depends on the tail of the distribution we are considering. Moreover, the two convergent series we obtain are defined precisely in the same regions of the distribution as those considered in the application of the saddlepoint method. Furthermore, the summed series that we derive for each of these regions correspond almost exactly with our saddlepoint approximations.

In the final section of the paper we report some brief numerical results to illustrate the performance of the saddlepoint approximation in relation to the interesting numerical comparisons between the Edgeworth approximation and the exact distribution that have already been carried out by Anderson and Sawa [**3**, **4**]. Our computations in this section suggest that the new approximation performs very well relative to the Edgeworth approximation. In all cases we have considered, the saddlepoint approximation is uniformly better in the tails of the distribution than the Edgeworth approximation (to $O(\mu^{-1})$ and to $O(\mu^{-3})$). Moreover, the saddlepoint approximation does well when the Edgeworth approximation (including that based on the first four terms, i.e., up to $O(\mu^{-3})$) does not, namely when the degree of overidentification in the equation is large.

2. THE MODEL AND ASSUMPTIONS

The model, notation and assumptions we will use for our main development will be based on that of Sawa [**28**] and closely related to that of Anderson [**1**] and Anderson and Sawa [**3**]. We consider the single structural equation:

$$(1) \qquad y_1 = \beta y_2 + Z_1\gamma_1 + u,$$

where y_1 and y_2 are vectors of T observations on two endogenous variables, Z_1 is a $T\times K_1$ matrix of observations on K_1 exogenous variables, and u is a vector of random disturbances. The structural coefficients are the scalar parameter β and the parameter vector γ_1. The reduced form for the two endogenous variables in (1) is given by

$$(2) \qquad Y = Z\pi + V$$

where $Y = [y_1 \vdots y_2]$, $Z = [Z_1 \vdots Z_2]$ is a $T\times K$ $(K = K_1 + K_2)$ matrix of exogenous variables, and $V = [v_1 \vdots v_2]$ is a matrix of reduced form disturbances. We partition π so that we can write (2) in the form

$$(3) \qquad \begin{aligned}[y_1 \vdots y_2] &= [Z_1 \vdots Z_2]\begin{bmatrix}\pi_{11} & \pi_{12}\\ \pi_{21} & \pi_{22}\end{bmatrix} + [v_1 \vdots v_2]\\ &= Z_1[\pi_{11} \vdots \pi_{12}] + Z_2[\pi_{21} \vdots \pi_{22}] + [v_1 \vdots v_2].\end{aligned}$$

We assume: that each row of $[v_1 \vdots v_2]$ is independently and identically distributed as a normal vector with zero mean and nonsingular covariance matrix

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix};$$

that equation (1) is identified by zero restrictions in the structural coefficients; that the observation matrix Z is nonstochastic, of rank K, and $T > K$; and that, for simplicity, $Z_1'Z_2 = 0$. The last assumption is a convention which is used by Sawa [**27**] and causes no real loss of generality. When it is not satisfied, equation (3) can readily be transformed so that it is true and this transformation will affect our results below only through the definition of the concentration parameter μ^2 (see (8) and the associated footnote).

We will concentrate on the k class estimator $\hat{\beta}_k$ of β in equation (1) which has the form (see Sawa [**28**, equation (2.11)])

$$\hat{\beta}_k = \frac{y_2'A_ky_1}{y_2'Aky_2}, \tag{4}$$

where

$$A_k = (1-k)\{I - Z_1(Z_1'Z_1)^{-1}Z_1' - Z_2(Z_2'Z_2)^{-1}Z_2'\} + Z_2(Z_2'Z_2)^{-1}Z_2. \tag{5}$$

We will confine ourselves to the case where k is nonstochastic.

As in Sawa [**28**] and Anderson and Sawa [**3**] we will work on the assumption that the covariance matrix Ω of the endogenous variables has been transformed to an identity matrix. This implies a transformation of (1) in which the parameter β is replaced by

$$\beta^* = \sqrt{\frac{\omega_{22}}{\omega_{11} - \omega_{12}^2/\omega_{22}}}\left(\beta - \frac{\omega_{12}}{\omega_{22}}\right) \tag{6}$$

(see Sawa [**26**, p. 657], and Anderson and Sawa [**3**, p. 692] where β^* is denoted by α). An equivalent transformation of (4), i.e.

$$\hat{\beta}_k^* = \sqrt{\frac{\omega_{22}}{\omega_{11} - \omega_{12}^2/\omega_{22}}}\left(\hat{\beta}_k - \frac{\omega_{12}}{\omega_{22}}\right), \tag{7}$$

gives us the k-class estimator of β^* in the transformed system. *In what follows, unless explicity stated, we will deal with the transformed system and, for convenience, drop the asterisk on* $\hat{\beta}_k^*$.

3. DERIVATION OF THE SADDLEPOINT APPROXIMATION TO THE DISTRIBUTION OF $\hat{\beta}_k$

We let $L(w_1, w_2)$ denote the Laplace transform of the joint density of $y_2'A_ky_2$ and $y_2'A_ky_1$ and note that, from Lemma 4 of Sawa [**28**, p. 664],[7] $L(w_1, w_2)$ is well defined for $1 - 2\text{Re}\,(w_1) - (\text{Re}\,(w_2))^2 > 0$ and $0 \leq k \leq 1$, where Re $(\cdot)$ denotes the

[7] $L(w_1, w_2)$ can alternatively be derived by writing $y_2'A_ky_2$ and $y_2'A_1y$ as quadratic forms in the vector (y_1', y_2') and using the expression for the characteristic function of a quadratic form in normal variates given in Lukacs and Laha [**17**, p. 55].

real part of a complex number, and is given by

$$L(w_1, w_2) = (1-2w_1-w_2^2)^{-K_2/2}(1-2hw_1-h^2w_2^2)^{-(T-K)/2} \cdot \exp\left[\frac{\mu^2}{2}\left\{\frac{(1+\beta w_2)^2}{1-2w_1-w_2^2}-1\right\}\right]$$

where[8]

(8) $$\mu^2 = \frac{\pi_{22}'(Z_2'Z_2)\pi_{22}}{\omega_{22}} \quad \text{and} \quad h = 1-k.$$

Since $y_2'A_ky_2 \geqslant 0$ (A_k is positive semi-definite when $0 \leqslant k \leqslant 1$; see Sawa [**28**, Lemma 2, p. 658]) and has a finite mean, the density of $\hat{\beta}_k$ can be obtained directly from the following formula for the density of a ratio:

(9) $$f(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left.\frac{\partial L(u-rw_2, w_2)}{\partial u}\right|_{u=0} dw_2$$

where c satisfies $1+2rc-c^2>0$. Cramér [**6**, p. 317] and Geary [**12**] gave the above formula for $f(r)$ when the path of integration is taken along the imaginary axis in the w_2 plane. It is easy to show that the path of integration in (9) is an allowable deformation when the stated condition on c is satisfied.[9]

Dropping the subscript on w_2, we find after a little manipulation that

(10) $$\left.\frac{\partial L(u-rw, w)}{\partial u}\right|_{u=0} = \left\{K_2(1+2hrw-h^2w^2)+h(T-K)(1+2rw-w^2) + \mu^2\frac{(1+2hrw-h^2w^2)(\beta w+1)^2}{(1+2rw-w^2)}\right\} \cdot (1+2rw-w^2)^{-(K_2+2)/2}(1+2hrw-h^2w^2)^{-(T-K+2)/2} \cdot \exp\left\{\frac{\mu^2}{2}\left(\frac{w^2(1+\beta^2)+2w(\beta-r)}{1+2rw-w^2}\right)\right\}$$

which we write in the form

(11) $$B(w)\exp\left\{\frac{\mu^2}{2}\Psi(w)\right\}$$

and then

(12) $$f(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} B(w)\exp\left\{\frac{\mu^2}{2}\Psi(w)\right\} dw.$$

The essence of the saddlepoint method is to select the path of integration in (12) in such a way that the major contribution to the value of the integral comes from the

[8] Note that when $Z_1'Z_2 \neq 0$, the definition of μ^2 should be changed to $\pi_{22}'Z_2'(I - Z_1(Z_1'Z_1)^{-1}Z_1')Z_2\pi_{22}/\omega_{22}$.

[9] Details are available in Appendix A of [**15**].

value of the integrand (1) in a region of a saddlepoint on the real axis. We will discuss later the question of the existence of a suitable saddlepoint and provide a further explanation of the saddlepoint method in Appendix A. However, we note now that in selecting the path of integration in (12) we must ensure that c satisfies the inequality

(13) $$1+2rc-c^2>0.$$

For, if (13) is satisfied, $(-rc, c)$ will lie in the region of (w_1, w_2) space where $L(w_1, w_2)$ is well defined. The inversion formula (12) will then be valid.

We see that (13) holds when c lies between the points $r \pm (1+r^2)^{\frac{1}{2}}$ on the real axis. The functions $B(w)$ and $\Psi(w)$ appearing in the integrand (11) are then analytic in the strip of the imaginary axis lying between the points $r \pm (1+r^2)^{\frac{1}{2}}$, i.e. for all complex w satisfying

(14) $$r-(1+r^2)^{\frac{1}{2}} < \mathrm{Re}\,(w) < r+(1+r^2)^{\frac{1}{2}}.$$

To obtain an approximate evaluation of the integral (12) we now select a path of integration in which $c = w^0$ is a suitable saddlepoint at which

(15) $$\Psi'(w^0) = 0.$$

We refer the reader to Appendix A for a discussion of why the solutions of (15) are saddlepoints. The path of integration in the vicinity of w^0 is now taken to be the curve along which the absolute magnitude of the intergrand of (12) decreases most rapidly (i.e., a curve of steepest descent from the saddlepoint). To find this curve we proceed as follows.

Expanding $\Psi(w)$ in a region of w^0 we have, in view of (15),

(16) $$\begin{aligned}\Psi(w) &= \Psi(w^0) + \frac{1}{2!}\Psi''(w^0)(w-w^0)^2 + O(\|w-w^0\|^3)\\ &= \Psi(w^0) + \frac{1}{2!}\Psi''(w^0)\{(x-x^0)^2 - y^2 + 2i(x-x^0)y\}\\ &\quad + O(((x-x^0)^2+y^2)^{\frac{3}{2}}).\end{aligned}$$

In the vicinity of $w^0 = x^0$, the behavior of $\Psi(w)$ is determined by the first two terms of (16). In the cases we consider below $\Psi''(w^0) > 0$, so the $\mathrm{Re}\,(\Psi(w))$ decreases most rapidly when $x = x^0$ and $y \neq 0$. Thus, in the vicinity of w^0, the curve of steepest descent becomes the straight line

(17) $$w = x^0 + iy$$

which crosses the real axis orthogonally at $w^0 = x^0$. We note also that, on the path defined by (17) and in the vicinity of x^0, $|\exp\{(\mu^2/2)\Psi(w)\}|$ is dominated by

(18) $$\exp\left\{\frac{\mu^2}{2}\Psi(w^0)\right\}\exp\left\{-\frac{\mu^2}{2}\Psi''(w^0)y^2\right\}.$$

As μ^2 becomes large it is clear that most of the contribution of this factor (i.e. of

$\exp\{(\mu^2/2)\,\Psi(w)\}$ to the value of the integrand in (12) arises in the immediate vicinity of the saddlepoint w^0.

The other factor in the integrand (12) is $B(w)$. We can see from its definition in (10) and (11) that, when $h=0$ (the case where $\hat{\beta}_k$ is the 2SLS estimator) or when $h\neq 0$ and T is fixed, the factor $B(w)$ is of $O(\mu^2)$ in a region of w^0. Hence, the contribution of this factor to the integrand as μ^2 becomes large is dominated by the behavior of (18) on the path of integration given in (17).

Taking w^0 to be a suitable saddlepoint, we now utilize the inversion formula (12) where the path of intergration corresponds to the lines of steepest descent through w^0, i.e. (17). Changing the variable of integration in (12) from w to y in $w=w^0+iy$, we have

$$(19)\qquad h(r)=\frac{1}{2\pi}\int_{-\infty}^{\infty} B(w^0+iy)\exp\left\{\frac{\mu^2}{2}\,\Psi(w^0+iy)\right\}dy.$$

Since $B(\cdot)$ and $\Psi(\cdot)$ are analytic on the path of integration, their Taylor expansions exist and can be utilized in (19). It can then be verified that (19) reduces to[10]

$$(20)\qquad \frac{B(w^0)\exp\left\{\frac{\mu^2}{2}\,\Psi(w^0)\right\}}{\pi^{\frac{1}{2}}\mu(\Psi''(w^0))^{\frac{1}{2}}}\left[1+\frac{1}{\mu^2}\left\{-\frac{B''(w^0)}{B(w^0)\Psi''(w^0)}+\frac{1}{4}\,\frac{\Psi^{(4)}(w^0)}{\Psi''(w^0)}\right.\right.$$
$$\left.\left.+\frac{\Psi^{(3)}(w^0)B'(w^0)}{(\Psi''(w^0))^2B(w^0)}-\frac{5}{12}\,\frac{(\Psi^{(3)}(w^0))^2}{(\Psi''(w^0))^3}\right\}+O(\mu^{-4})\right].$$

The first factor in (20) is the saddlepoint approximation and the series is sometimes referred to as the saddlepoint expansion. The justification of the expansion as a valid asymptotic series (in which the remainder has the same order of magnitude in $1/\mu^2$ as the first neglected term) is obtained, as in Daniels [**7**], by the use of Watson's Lemma (see [**7**, p. 633]).

We now proceed to evaluate (20) in detail. First we must locate the appropriate saddlepoint w^0 and verify that $\Psi''(w^0)>0$. Since w^0 satisfies (15), we derive

$$(21)\qquad \Psi'(w)=\frac{2\beta(\beta r+1)(w-w_1^0)(w-w_2^0)}{(1+2rw-w^2)^2}$$

where $w_1^0=(r-\beta)/(1+\beta r)$ and $w_2^0=-1/\beta$. Differentiating (21) again we obtain

$$(22)\qquad \Psi''(w_1^0)=\frac{2(\beta r+1)^4}{(1+2\beta r-\beta^2)(r^2+1)^2}$$

and

$$(23)\qquad \Psi''(w_2^0)=\frac{2\beta^4}{\beta^2-2r\beta-1}.$$

[10] Full details of the derivation are given in [**15**] which is available on request.

The signs of $\Psi''(w_1^0)$ and $\Psi''(w_2^0)$ both depend on the sign of $\beta^2-2r\beta-1$. We can distinguish the cases given in Table I. It is clear from the table that, for many values of β, we will need to utilize a different saddlepoint at different regions of the distribution. Thus, when $\beta=1$, we deform the path of integration to pass through the saddlepoint $w_1^0=(r-1)/(r+1)$ for $r>0$ and through the saddlepoint $w_2^0=-1$ for $r<0$. We then obtain a different approximation for each tail of the distribution of $\hat{\beta}_k$. On the other hand, when β is close to zero but still positive, the inequality $r>(\beta^2-1)/2\beta$ will hold for most values of interest of the argument r and in this case we need only be concerned with the saddlepoint w_1^0.

TABLE I

Sign of $\beta^2-2\beta r-1$	Region of distribution	Saddlepont w^0	Sign of $\Psi''(\ )$
	$\beta>0$		
$-$ve	$r>\dfrac{\beta^2-1}{2\beta}$	$w_1^0=\dfrac{r-\beta}{\beta r+1}$	$\Psi''(w_1^0)>0$
$+$ve	$r<\dfrac{\beta^2-1}{2\beta}$	$w_2^0=-\dfrac{1}{\beta}$	$\Psi''(w_2^0)>0$
	$\beta<0$		
$-$ve	$r<\dfrac{\beta^2-1}{2\beta}$	$w_1^0=\dfrac{r-\beta}{\beta r+1}$	$\Psi''(w_1^0)>0$
$+$ve	$r>\dfrac{\beta^2-1}{2\beta}$	$w_2^0=-\dfrac{1}{\beta}$	$\Psi''(w_2^0)>0$

Our earlier argument leading to the expansion (20) relied on the functions $\Psi(w)$ and $B(w)$ being analytic in a strip of the imaginary axis containing w^0. We know that $\psi(w)$ and $B(w)$ are analytic in the region of the complex plane defined by (14). Hence, for the validity of (20), we require that the saddlepoint w^0 (which is, in both cases, real) satisfy the inequality (14). In fact w^0 will satisfy (14) if and only if

$$1+2rw^0-w^{02}>0. \tag{24}$$

For w_1^0, we find that the left side of (24) is $(1+r^2)(1+2\beta r-\beta^2)/(\beta r+1)^2$ which is positive when $1+2\beta r-\beta^2>0$. Referring to Table I, this will be true for $r>(\beta^2-1)/2\beta$ when $\beta>0$, i.e. in the right tail of the distribution. For w_2^0 we find that the left side of (24) is $-(1+2\beta r-\beta^2)/\beta^2$ which is positive when $1+2\beta r-\beta^2<0$. Referring to Table I again, we see that this is true for $r<(\beta^2-1)/2\beta$ when $\beta>0$, i.e. in the left tail of the distribution. Thus, our condition on the selection of the saddlepoint (which depends on the sign of $\beta^2-2\beta r-1$ corresponds with the condition under which the saddlepoint will lie within the region of the complex plane defined in (14). Once we have selected the appropriate saddlepoint according to the sign of $\beta^2-2\beta r-1$, we can therefore be sure that this saddlepoint will lie within the region in which $\Psi(w)$ and $B(w)$ are analytic.

To specify the first factor of (20) we now need only evaluate $B(w^0)$ and $\Psi(w^0)$. After some manipulation we find that *in the case of* $\beta>0$ the saddlepoint approximation is given by[11]

$$h(r)=\frac{1}{\sqrt{2\pi\mu}}\{K_2(\beta^2-2h\beta r-h^2)+h(T-K)(\beta^2-2\beta r-1)\}$$
$$\cdot\beta^{T-K_1}(\beta^2-2\beta r-1)^{-(K_2+1)/2}(\beta^2-2h\beta r-h^2)^{-(T-K+2)/2}e^{-\mu^2/2}$$

when

$$r<\frac{\beta^2-1}{2\beta},\quad \beta>0,$$

and

$$\begin{aligned}h(r)=\frac{1}{\sqrt{2\pi\mu}}&[K_2\{(\beta r+1)^2+2rh(r-\beta)(\beta r+1)-h^2(r-\beta)^2\}\\&+h(T+K)(1+2\beta r-\beta^2)(r^2+1)\\&+\mu^2\{(\beta r+1)^2+2rh(r-\beta)(\beta r+1)-h^2(r-\beta)^2\}(1+2\beta r-\beta^2)\\&\cdot(r^2+1)^{-1}](\beta r+1)^{T-K_1}(1+2\beta r-\beta^2)^{-(K_2+1)/2}(r^2+1)^{-K_2/2}\\&\cdot\{(\beta r+1)^2+2rh(r-\beta)(\beta r+1)-h^2(r-\beta)^2\}^{-(T-K+2)/2}\\&\cdot\exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{r^2+1}\right\}\end{aligned}$$

when

$$r>\frac{\beta^2-1}{2\beta},\quad \beta>0.$$

Similar formulae can be readily obtained when $\beta<0$. Note that the approximation $h(r)$ involves only elementary functions and can be readily computed numerically.

[11] Full details of the derivations are available in [**15**].

4. COMPARISON WITH THE EXACT DENSITY FUNCTION IN THE CASE OF 2SLS

The case of 2SLS corresponds to $h=0$. Hence, in that case, the saddlepoint approximation is just

$$
(25)\qquad h(r)=\begin{cases}\dfrac{1}{\sqrt{2\pi u}}K_2\beta^{K_2}(\beta^2-2\beta r-1)^{-(K_2+1)/2}\,e^{-\mu^2/2} \\[1ex] \qquad \text{when}\quad r<\dfrac{\beta^2-1}{2\beta},\quad \beta>0; \\[1ex] \dfrac{1}{\sqrt{2\pi u}}\left[K_2+\mu^2\dfrac{1+2\beta r-\beta^2}{1+r^2}\right](\beta r+1)^{K_2}(1+2\beta r-\beta^2)^{-(K_2+1)/2} \\[1ex] \qquad \cdot(1+r^2)^{-K_2/2}\exp\left\{-\dfrac{\mu^2}{2}\dfrac{(r-\beta)^2}{1+r^2}\right\} \\[1ex] \qquad \text{when}\quad r>\dfrac{\beta^2-1}{2\beta},\quad \beta>0.\end{cases}
$$

Richardson [25] has shown that the exact probability density function of the transformed 2SLS estimator (see (7)) is given, in our notation, by

$$
(26)\qquad f(r)=\frac{1}{B\left(\frac{1}{2},\frac{K_2}{2}\right)}e^{-(\mu^2/2)(1+\beta^2)}(1+r^2)^{-(K_2+1)/2}
$$
$$
\cdot\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{K_2+1}{2}+j\right)\Big/\Gamma\left(\frac{K_2+1}{2}\right)}{\Gamma\left(\frac{K_2}{2}+j\right)\Big/\Gamma\left(\frac{K_2}{2}\right)}\frac{1}{j!}\left(\frac{\mu^2}{2}\frac{(1+\beta r)^2}{1+r^2}\right)^j
$$
$$
\cdot{}_1F_1\left(\frac{K_2-1}{2},j+\frac{K_2}{2},\frac{\mu^2\beta^2}{2}\right)
$$

where $B(a,b)$ is the beta function and ${}_1F_1(a,b,x)$ is the confluent hypergeometric function.

We will use later in this paper an alternative expression for $f(r)$ which is obtained as follows. We can write the summation in (26) as

$$
\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma\left(\frac{K_2+1}{2}+j\right)\Big/\Gamma\left(\frac{K_2+1}{2}\right)}{\Gamma\left(\frac{K_2}{2}+j\right)\Big/\Gamma\left(\frac{K_2}{2}\right)}\cdot\frac{\Gamma\left(\frac{K_2-1}{2}+l\right)\Gamma\left(\frac{K_2-1}{2}\right)}{\Gamma\left(\frac{K_2}{2}+j+l\right)\Gamma\left(\frac{K_2}{2}+j\right)}
$$
$$
\cdot\frac{1}{j!}\frac{1}{l!}\left(\frac{\mu^2}{2}\frac{(1+\beta r)^2}{1+r^2}\right)^j\left(\frac{\mu^2\beta^2}{2}\right)^l
$$

and interchanging the order of summation we obtain

$$(27)\qquad \sum_{l=0}^{\infty} \frac{\Gamma\left(\frac{K_2-1}{2}+l\right)\Big/\Gamma\left(\frac{K_2-1}{2}\right)}{\Gamma\left(\frac{K_2}{2}+l\right)\Big/\Gamma\left(\frac{K_2}{2}\right)} \frac{1}{l!}\left(\frac{\mu^2\beta^2}{2}\right)^l {}_1F_1\left(\frac{K_2+1}{2}, \frac{K_2}{2}+l, \frac{\mu^2}{2}\frac{(1+\beta r)^2}{1+r^2}\right).$$

We will also use the fact that if $x>0$ and b is neither zero nor a negative integer, then ${}_1F_1(a, b, x)$ has the following asymptotic expansion, as $x\to\infty$ (see Lebedev [**16**, pp. 268–271] and Sawa [**28**, p. 667])[12]

$$(28)\qquad {}_1F_1(a, b, x)=\frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}[1+(b-a)(1-a)x^{-1}+O(x^{-2})].$$

Now consider (26) where

$${}_1F_1\left(\frac{K_2-1}{2}, j+\frac{K_2}{2}, \frac{\mu^2\beta^2}{2}\right)$$

is replaced by the asymptotic expansion (28). We obtain, after some simplification,

$$(29)\qquad f(r)=\frac{K_2-1}{\sqrt{2\pi}\mu\beta} e^{-\mu^2/2}(1+r^2)^{-(K_2+1)/2} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{K_2+1}{2}+j\right)}{\Gamma\left(\frac{K_2+1}{2}\right)j!}\left[\frac{(1+\beta r^2)}{\beta^2(1+r^2)}\right]^j$$
$$\cdot\left[1-\left(j+\frac{1}{2}\right)\left(\frac{K_2-3}{2}\right)\left(\frac{\mu^2\beta^2}{2}\right)^{-1}+O(\mu^{-4})\right].$$

In order to use the negative binomial expansion $(l-x)^{-m}=\Sigma_{j=0}^{\infty}(\Gamma(m+j)/(j!\Gamma(m))x^j$, which is valid if $|x|<1$ we require that

$$(30)\qquad \frac{(1+\beta r)^2}{\beta^2(1+r^2)}<1 \quad\text{or}\quad \beta^2-2\beta r-1>0.$$

If (30) is satisfied, then the terms on the expansion (29) sum to

$$(31)\qquad \frac{K_2-1}{\sqrt{2\pi}\,\mu}\beta^{K_2} e^{-\mu^2/2}(\beta^2-2\beta r-1)^{-(K_2+1)/2}$$
$$\cdot\left[1-\left(\frac{K_2-3}{2}\right)\left(\frac{1}{\mu^2\beta^2}\right)\left\{1+\frac{K_2+1}{2}\frac{(1+\beta r)^2}{\beta^2-2\beta r-1}\right\}+O(\mu^{-4})\right].$$

[12] Higher order terms in this expansion are given explicitly by Lebedev and Sawa and can readily be used in what follows to obtain higher order terms in the corresponding expansion of the exact density.

Note that the dominant term of (31) differs from the saddlepoint approximation (25) only by the constant coefficient K_2-1.

When we have $1+2\beta r-\beta^2>0$ we use the alternative summation (27) in order to obtain an expansion for (26). If we replace, in (27),

$$ {}_1F_1\left(\frac{K_2+1}{2}, \frac{K_2}{2}+l, \frac{\mu^2}{2}\frac{(1+\beta r)^2}{1+r^2}\right) $$

by its asymptotic expansion, we obtain after simplifications

$$ f(r)=\frac{\mu}{\sqrt{2\pi}}\left(\frac{(1+\beta r)^2}{1+r^2}\right)^{\frac{1}{2}}\exp-\left(\frac{\mu^2}{2}\frac{(r-\beta)^2}{1+r^2}\right)(1+r^2)^{-(K_2+1)/2} \tag{32} $$

$$ \cdot\sum_{l=0}^{\infty}\frac{\Gamma\left(\frac{K_2-1}{2}+l\right)}{\Gamma\left(\frac{K_2-1}{2}\right)l!}\left(\frac{(1+\beta r)^2}{\beta^2(1+r^2)}\right)^{-l}\left[1+\left(\frac{K_2-1}{2}\right)\frac{1+r^2}{\mu^2(1+\beta r)^2}\right] $$

$$ -\sum_{l=0}^{\infty}\frac{\Gamma\left(\frac{K_2-1}{2}+l\right)}{\Gamma\left(\frac{K_2-1}{2}\right)(l-1)!}\left(\frac{K_2-1}{2}\right) $$

$$ \cdot\left(\frac{\mu^2}{2}\frac{(1+\beta r)^2}{1+r^2}\right)^{-1}\left(\frac{(1+\beta r^2}{\beta^2(1+r^2)}\right)^{-l}+O(\mu^{-4}). $$

When $1+2\beta r-\beta^2>0$ we see that $\beta^2(1+r^2)/(1+\beta r)^2<1$ and the series in (32) can be summed by using the negative binomial expansion. After some manipulation we obtain

$$ f(r)=\frac{\mu}{\sqrt{2\pi}}(1+\beta r)^{K_2}(1+r^2)^{-(K_2+2)/2}(1+2\beta r-\beta^2)^{-(K_2-1)/2} \tag{33} $$

$$ \cdot\exp\left(-\frac{\mu^2}{2}\frac{(r-\beta)^2}{1+r^2}\right)\cdot\left\{1+\frac{1}{\mu^2}\left(\frac{K_2-1}{2}\right)\left(\frac{1+r^2}{(1+\beta r)^2}\right)\right. $$

$$ \left.\cdot\left[1-\frac{(K_2-1)\beta^2(1+r^2)}{(1+\beta r)^2(1+2\beta r-\beta^2)}\right]+O(\mu^{-4})\right\}. $$

It should be noted that (44) is equal to the dominant term (i.e., the term with a coefficient order of μ) in the saddlepoint approximation in the case where $1+2\beta r-\beta^2>0$. The relative difference between (33) and the saddlepoint approximation in the right hand tail is of order μ^{-2}.

5. BRIEF NUMERICAL RESULTS

We examine the performance of the saddlepoint approximation for the 2SLS estimate in three cases considered by Anderson and Sawa [3] and in one heavily overidentified case ($K_2 = 20$). We report our numerical computations in Tables II, III, IV, and V. We give the ordinates of the exact density,[13] the Edgeworth approximation up to $O(\mu^{-1})$ and $O(\mu^{-3})$, and the ordinates of the saddlepoint approximation (25). The new density approximations have been renormalized so that the area under the curve is unity. For the parameter values we have considered, we find that $1+2\beta r-\beta^2>0$ for all values of r of interest and the formulae for the extreme left hand tail of the distribution are not needed.

TABLE II

		$\beta=0.6,\ K_2=4,\ \mu^2=80$			
$r-\beta$	Exact Density	A–S to $0(\mu^{-1})$	A–S to $O(\mu^{-3})$	Saddlepoint	
1.00	0.0000	0.0000	0.0000	0.0000	a,b
0.90	0.0000	0.0000	0.0000	0.0000	a,b
0.80	0.0001	0.0000	0.0000	0.0001	a,b
0.70	0.0004	0.0000	0.0002	0.0004	a,b
0.60	0.0020	0.0004	0.0019	0.0020	a,b
0.50	0.0099	0.0062	0.0120	0.0100	a,b
0.40	0.0476	0.0492	0.0539	0.0478	a,b
0.30	0.2049	0.2253	0.2212	0.2054	a,b
0.24	0.4498	0.4663	0.4739	0.4504	a,b
0.20	0.7219	0.7231	0.7469	0.7225	a,b
0.18	0.8981	0.8900	0.9214	0.8985	a,b
0.14	1.3336	1.3107	1.3500	1.3336	a,b
0.10	1.8603	1.8363	1.8682	1.8596	a,b
0.08	2.1401	2.1211	2.1445	2.1389	a,b
0.06	2.4153	2.4035	2.4172	2.4137	a,b
0.04	2.6708	2.6664	2.6712	2.6688	a
0.02	2.8902	2.8912	2.8900	2.8878	
0.00	3.0567	3.0597	3.0564	3.0542	a
−0.02	3.1554	3.1567	3.1551	3.1529	
−0.04	3.1753	3.1718	3.1745	3.1729	a
−0.06	3.1108	3.1012	3.1089	3.1088	a
−0.08	2.9632	2.9484	2.9593	2.9617	a,b
−0.10	2.7409	2.7239	2.7343	2.7401	a,b
−0.14	2.1370	2.1277	2.1231	2.1374	a,b
−0.18	1.4605	1.4698	1.4393	1.4619	a,b
−0.20	1.1461	1.1639	1.1221	1.1478	a,b
−0.24	0.6331	0.6582	0.6064	0.6348	a,b
−0.30	0.1960	0.2084	0.1749	0.1970	a,b
−0.40	0.0128	0.0061	0.0089	0.0129	a,b
−0.50	0.0003	−0.0022	−0.0002	0.0003	a,b
−0.60	0.0000	−0.0003	−0.0004	0.0000	a,b
−0.70	0.0000	−0.0000	−0.0001	0.0000	a,b
−0.80	0.0000	−0.0000	−0.0000	0.0000	a,b
−0.90	0.0000	−0.0000	−0.0000	0.0000	a,b
−1.00	0.0000	−0.0000	−0.0000	0.0000	a,b

[a] Saddlepoint approximation as close or closer to exact density than A–S to $O(\mu^{-1})$.

[b] Saddlepoint approximation as close or closer to exact density than A–S to $O(\mu^{-3})$.

A-S Edgeworth approximation derived by Anderson and Sawa [3].

[13] In an earlier draft of this article [**15**] we used the exact density ordinates given in the article by Anderson and Sawa [**3**]. The further computations we have carried out for this version of our article indicate that the ordinates given in [**3**] are in error. The exact density ordinates as well as the ordinates of the Edgeworth approximation detailed in Tables II–V are based on our own computations.

TABLE III

$r-\beta$	Exact Density	$\beta=0.6$, $K_2=10$, $\mu^2=80$ A–S to $O(\mu^{-1})$	A–S to $O(\mu^{-3})$	Saddlepoint	
1.00	0.0000	0.0000	0.0000	0.0000	a,b
0.90	0.0000	0.0000	0.0000	0.0000	a,b
0.80	0.0000	0.0000	0.0000	0.0000	a,b
0.70	0.0001	0.0000	0.0001	0.0001	a,b
0.60	0.0003	0.0003	0.0010	0.0004	a,b
0.50	0.0029	0.0036	0.0093	0.0024	a,b
0.40	0.0145	0.0199	0.0650	0.0148	a,b
0.30	0.0826	0.0531	0.2628	0.0835	a,b
0.24	0.2143	0.1091	0.4610	0.2155	a,b
0.20	0.3845	0.2236	0.6282	0.3856	a,b
0.18	0.5057	0.3278	0.7297	0.5065	a,b
0.14	0.8389	0.6736	0.9943	0.8383	a,b
0.10	1.3062	1.2327	1.3776	1.3030	a,b
0.08	1.5871	1.5844	1.6210	1.5820	a,b
0.06	1.8911	1.9663	1.8952	1.8840	a
0.04	2.2071	2.3573	2.1905	2.1979	a,b
0.02	2.5199	2.7311	2.4911	2.5087	a,b
0.00	2.8105	3.0597	2.7763	2.7979	a,b
−0.02	3.0584	3.3168	3.0217	3.0449	a,b
−0.04	3.2426	3.4809	3.2028	3.2291	a,b
−0.06	3.3451	3.5383	3.2981	3.3328	a,b
−0.08	3.3533	3.4852	3.2928	3.3432	a,b
−0.10	3.2622	3.3274	3.1815	3.2553	a,b
−0.14	2.8076	2.7648	2.6724	2.8085	a,b
−0.18	2.1118	2.0319	2.9243	2.1199	a,b
−0.20	1.7365	1.6634	1.5315	1.7470	a,b
−0.24	1.0503	1.0154	0.8358	1.0624	a,b
−0.30	0.3699	0.3805	0.1992	0.3784	a,b
−0.40	0.0293	0.0354	−0.0259	0.0309	a,b
−0.50	0.0009	0.0004	−0.0060	0.0010	a,b
−0.60	0.0000	−0.0001	−0.0004	0.0000	a,b
−0.70	0.0000	−0.0000	−0.0000	0.0000	a,b
−0.80	0.0000	−0.0000	−0.0000	0.0000	a,b
−0.90	0.0000	−0.0000	−0.0000	0.0000	a,b
−1.00	0.0000	−0.0000	−0.0000	0.0000	a,b

[a] Saddlepoint approximation as close or closer to exact density than A–S to $O(\mu^{-1})$.
[b] Saddlepoint approximation as close or closer to exact density than A–S to $O(\mu^{-3})$.
A-S Edgeworth approximation derived by Anderson and Sawa [3].

From the ratings indicated in the final column of the tables it is clear that the saddlepoint approximation performs very well compared with the Edgeworth approximation. It is uniformly better in the tails than the Edgeworth for all parameter values considered and in the heavily overidentified case (Table V) it outperforms the Edgeworth over virtually the whole range of the distribution. In the latter case, we note that the errors involved in the Edgeworth approximation are so substantial that this approximation gives no reliable guide to the shape of the density in any region of the distribution. To the extent that the saddlepoint approximation is a good approximation in this case as well as the others it would

TABLE IV

$\beta=0.6,\ K_2=4,\ \mu^2=40$

$r-\beta$	Exact Density	A–S to $0(\mu^{-1})$	A–S to $O(\mu^{-3})$	Saddlepoint	
1.00	0.0007	0.0000	0.0002	0.0008	a,b
0.90	0.0017	0.0001	0.0014	0.0018	a,b
0.80	0.0041	0.0010	0.0056	0.0043	a,b
0.70	0.0103	0.0063	0.0160	0.0106	a,b
0.60	0.0263	0.0269	0.0366	0.0268	a,b
0.50	0.0669	0.0832	0.0845	0.0678	a,b
0.40	0.1664	0.1951	0.2024	0.1677	a,b
0.30	0.3911	0.3966	0.4421	0.3923	a,b
0.24	0.6241	0.6028	0.6690	0.6248	a,b
0.20	0.8304	0.7961	0.8648	0.8303	a,b
0.18	0.9491	0.9117	0.9772	0.9484	a,b
0.14	1.2136	1.1787	1.2295	1.2116	a,b
0.10	1.5031	1.4799	1.5093	1.4994	a,b
0.08	1.6503	1.6348	1.6533	1.6458	a
0.06	1.7942	1.7862	1.7951	1.7889	a
0.04	1.9304	1.9285	1.9301	1.9243	
0.02	2.0539	2.0562	2.0531	2.0473	
0.00	2.1597	2.1636	2.1588	2.1526	
−0.02	2.2428	2.2455	2.2419	2.2356	
−0.04	2.2988	2.2980	2.2975	2.2916	
−0.06	2.3238	2.3178	2.3218	2.3171	
−0.08	2.3154	2.3037	2.3119	2.3093	a
−0.10	2.2724	2.2554	2.2666	2.2672	a,b
−0.14	2.0861	2.0649	2.0746	2.0835	a,b
−0.18	1.7892	1.7753	1.7675	1.7896	a,b
−0.20	1.6131	1.6068	1.5860	1.6148	a,b
−0.24	1.2393	1.2521	1.2015	1.2434	a,b
−0.30	0.7211	0.7553	0.6697	0.7267	a,b
−0.40	0.1956	0.2164	0.1464	0.1994	a,b
−0.50	0.0326	0.0263	0.0121	0.0339	a,b
−0.60	0.0036	−0.0052	0.0005	0.0038	a,b
−0.70	0.0003	−0.0031	−0.0010	0.0003	a,b
−0.80	0.0000	−0.0007	−0.0013	0.0000	a,b
−0.90	0.0000	−0.0001	−0.0006	0.0000	a,b
−1.00	0.0000	−0.0000	−0.0001	0.0000	a,b

[a] Saddlepoint approximation as close or closer to exact density than A–S to $O(\mu^{-1})$.
[b] Saddlepoint approximation as close or closer to exact density than A–S to $O(\mu^{-3})$.
A-S Edgeworth approximation derived by Anderson and Sawa [3].

appear that this approximation is less sensitive to parameter variations in terms of its performance than the Edgeworth approximation. Given that many equations in large macromodels are heavily overidentified this is a useful feature of the new approximation.

It may also be worth drawing attention to the numerical accuracy of the saddlepoint approximation in the cases considered. From Table II we see that this approximation gives three decimal place accuracy over a substantial part of the tail regions. The improvement over the Edgeworth approximation in this respect is very substantial.

TABLE V

		$\beta=1.0,\ K_2=20,\ \mu^2=100$			
$r-\beta$	Exact Density	A-S to $O(\mu^{-1})$	A-S to $O(\mu^{-3})$	Saddlepoint	
1.00	0.0000	0.0000	0.0000	0.0000	a,b
0.90	0.0000	0.0000	0.0000	0.0000	a,b
0.80	0.0000	0.0000	0.0000	0.0000	a,b
0.70	0.0000	0.0000	0.0012	0.0000	a,b
0.60	0.0000	0.0000	0.0194	0.0000	a,b
0.50	0.0002	−0.0061	0.1791	0.0002	a,b
0.40	0.0017	−0.0827	0.8584	0.0018	a,b
0.30	0.0121	−0.4386	2.1160	0.0123	a,b
0.24	0.0369	−0.7849	2.6281	0.0368	a,b
0.20	0.0748	−0.9340	2.6212	0.0738	a,b
0.18	0.1050	−0.9339	2.4901	0.1032	a,b
0.14	0.2011	−0.6937	1.9961	0.1963	a,b
0.10	0.3678	−0.0549	1.3157	0.3570	a,b
0.08	0.4879	0.4154	0.9782	0.4726	a,b
0.06	0.6380	0.9678	0.6894	0.6170	a,b
0.04	0.8216	1.5763	0.4830	0.7936	a,b
0.02	1.0410	2.2069	0.3849	1.0047	a,b
0.00	1.2960	2.8209	0.4090	1.2505	a,b
−0.02	1.5837	3.3788	0.5530	1.5285	a,b
−0.04	1.8971	3.8443	0.7969	1.8325	a,b
−0.06	2.2249	4.1885	1.1050	2.1522	a,b
−0.08	2.5513	4.3923	1.4305	2.4731	a,b
−0.10	2.8566	4.4488	1.7230	2.7768	a,b
−0.14	3.3149	4.1510	2.0350	3.2475	a,b
−0.18	3.4361	3.4438	1.8416	3.4041	a,b
−0.20	3.3405	3.0095	1.5714	3.3327	a,b
−0.24	2.8560	2.1216	0.8436	2.8992	a,b
−0.30	1.7154	1.0332	−0.1483	1.8043	a,b
−0.40	0.3170	0.1860	−0.4900	0.3662	a,b
−0.50	0.0176	0.0170	−0.1553	0.0239	b
−0.60	0.0003	0.0007	−0.0196	0.0005	a,b
−0.70	0.0000	−0.0000	−0.0012	0.0000	a,b
−0.80	0.0000	−0.0000	−0.0000	0.0000	a,b
−0.90	0.0000	−0.0000	−0.0000	0.0000	a,b
−1.00	0.0000	−0.0000	−0.0000	0.0000	a,b

[a] Saddlepoint approximation as close or closer to exact density than A-S to $O(\mu^{-1})$.
[b] Saddlepoint approximation as close or closer to exact density than A-S to $O(\mu^{-3})$.
A-S Edgeworth approximation derived by Anderson and Sawa [3].

6. FINAL COMMENTS

For the parameter values used in the computations of the last section only one of the approximating formulae in (25) was used. A problem arises in cases where we may wish to use both formulae. For, when $1+2\beta r-\beta^2=0$ neither formula applies and the integrand of (12) has a singularity at the point $(r-\beta)/(1+\beta r)=-1/\beta$ on the real axis. In other cases, both $(r-\beta)/(1+\beta r)$ and $-1/\beta$ are saddlepoints, as we have seen, and the integrand of (12) has a branch point and essential singularity at the point $r-(1+r^2)^{\frac{1}{2}}$ on the real axis between these two saddlepoints.[14] When r takes on values for which $1+2\beta r-\beta^2$ tends to zero, the two saddlepoints converge and the behavior of the integrand is very irregular in this region of the real axis (there is a path of steepest descent through $-1/\beta$ along the real axis and a path of steepest descent through $(r-\beta)/(1+\beta r)$ orthogonal to the real axis). When the saddlepoints coincide, they coalesce with the singularity at $r-(1+r^2)^{\frac{1}{2}}$ and it is clear that the approximations cannot hold uniformly for values of r in this neighborhood. A transitional form or uniform asymptotic approximation (see, for example, Bleistein et al. [**5**] and Phillips [**24**]) is needed to secure a good approximation in this region of the distribution.

The approximation to the density of the 2SLS estimator obtained in Section 4 is related to the expansions derived by Sargan [**26**] of the exact density of the instrumental variable (IV) estimator in the case of an equation with n endogenous variables (see Appendix B of [**26**]). Sargan reduces the joint density of the IV estimator of the coefficients of the included endogenous variables to the integral over a matrix space of a function involving a confluent hypergeometric function with a single matrix argument.[15] The latter function has an asymptotic expansion similar in form to (28) and when the dominant term of this expansion is used the integral can be readily calculated. The resulting approximation given by Sargan (equation (B14) in [**26**]) is then similar to our (31) and (33).

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APPENDIX A

SADDLEPOINTS AND CURVES OF STEEPEST DESCENT

For those readers not familiar with the saddlepoint technique we outline here some of the basic principles of the method. Further details may be obtained from the references [**5**, **7**, **9**, and **10**].

We start with equation (15) in the body of the paper, viz. $\Psi'(w^0)=0$. Solutions of this equation are called saddlepoints because if we write $w=x+iy$ and $\Psi(w)=a(x,y)+ib(x,y)$ then (x^0, y^0) is a saddlepoint of the function $a(x, y)$ and hence of the function

$$\left|\exp\left\{\frac{\mu^2}{2}\Psi(w)\right\}\right|=\exp\left\{\frac{\mu^2}{2}\operatorname{Re}(\Psi(w))\right\}=\exp\left\{\frac{\mu^2}{2}a(x,y)\right\}. \tag{A.1}$$

To see this we note that, since $\Psi(w)$ is analytic, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy the Cauchy Riemann conditions:

$$\frac{\partial a}{\partial x}=\frac{\partial b}{\partial y},\qquad \frac{\partial a}{\partial y}=-\frac{\partial b}{\partial x}, \tag{A.2}$$

[14] It may also be of interest to point out that by computing the residue at the singularity or by looping the integration path around the singularity (depending on the value of K_2) it is possible to extract the exact density. Details are given in [**23**].

[15] A non-integral expression for the density on this case is obtained in [**22**].

so that

$$\frac{\partial^2 a}{\partial x^2} = \frac{\partial^2 b}{\partial x \partial y}, \qquad \frac{\partial^2 a}{\partial y^2} = -\frac{\partial^2 b}{\partial y \partial x},$$

and hence, from the continuity of the second derivatives of b,

$$\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} = 0 \tag{A.3}$$

for all values of x and y. Since the first derivatives of $a(\cdot, \cdot)$ are zero at (x^0, y^0) it now follows from (A.3) that

$$\left(\frac{\partial^2 a}{\partial x^2}\right)\left(\frac{\partial^2 a}{\partial y^2}\right) - \left(\frac{\partial^2 a}{\partial x \partial y}\right) < 0$$

and (x^0, y^0) is a saddlepoint of $a(x, y)$.

It also follows from (A.2) that the gradient vectors $(\partial a/\partial x, \partial a/\partial y)$ and $(\partial b/\partial x, \partial b/\partial y)$ of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are orthogonal. This means that a normal vector to the curve $a(x, y) = \text{const.}$ is tangential to a member of the family of curves $\{b(x, y) = \text{const.}\}$. We can deduce that the curve of steepest descent along which $a(x, y)$, and hence (A.1), decreases most rapidly on either side of the saddlepoint w^0 is such that

$$\operatorname{Im}(\Psi(w)) = \operatorname{Im}(\Psi(w^0))$$

where $\operatorname{Im}(\cdot)$ denotes the imaginary part. That is,

$$b(x, y) = b(x^0, y^0). \tag{A.4}$$

In the present paper we find that the suitable saddlepoint w^0 is real, so that $\Psi(w^0)$ is also real and (A.4) becomes

$$b(x, y) = 0. \tag{A.5}$$

Since (A.1) decreases most rapidly as we move away from the saddlepoint w^0 along the curve defined by (A.5) and since (A.1) dominates the value of the integrand, at least as μ^2 becomes large, we find that the value of the integrand at the saddlepoint provides the dominant contribution to the value of the integral if we select this path of integration through the saddlepoint. The integral itself can now be evaluated approximately by expanding the components of the integrand in the Taylor series about the saddlepoint. The dominant contribution then comes out of the integral as a multiplicative factor, giving an asymptotic expansion of the integral in terms of $1/\mu^2$ of the form of (20) in the body of the paper.

APPENDIX B

An Approximation to the Distribution Function

We demonstrate here how a saddlepoint approximation can be derived for the distribution function rather than the probability density of $\hat{\beta}_k$. The method is quite general and can be used in other applications of the saddlepoint approximation.

From the work of Gurland [**14**] and Gil-Pelaez [**13**] we know that an inversion formula for the distribution function can be obtained by using the Cauchy principal value of an integral involving the characteristic function. Gurland has also considered ratios of random variables. In particular, from his Theorem 1 [**14**, p. 228], the distribution function, $F(x)$, of X_1/X_2 where $P(X_2 \leqslant 0) = 0$ satisfies

$$F(x) + F(x-0) = 1 - \frac{1}{\pi i} \oint \frac{\phi(t, -tx)}{t}\, dt$$

where $\phi(\cdot, \cdot)$ is the joint characteristic function of (X_1, X_2) and $\oint$ denotes the Cauchy principal value $\lim_{\varepsilon \to 0, R \to \infty} \{\int_\varepsilon^R + \int_{-R}^{-\varepsilon}\}$. Applying this result in the case of the distribution of the k-class estimator $\hat{\beta}_k$ we have

$$F(r)+F(r-0)=1-\frac{1}{\pi i}\oint\frac{L(-wr,\,w)}{w}\,dw \tag{B.1}$$

where, from Section 3 of the paper,

$$L(-wr,\,w)=C(w)\exp\left\{\frac{\mu^2}{2}\,\Psi(w)\right\} \tag{B.2}$$

where

$$C(w)=(1+2rw-w^2)^{-K_2/2}(1+2hrw-h^2w^2)^{-(T-K)/2},$$

$$\Psi(w)=(1+2rw-w^2)^{-1}\{w^2(1+\beta^2)+2w(\beta-r)\},$$

and the integration in (B.1) is now taken along the corresponding paths on the imaginary axis, i.e., $\lim_{\varepsilon\to0,R\to\infty}\{\int_{i\varepsilon}^{iR}+\int_{-iR}^{-i\varepsilon}\}$.

The integrand in (B.1) has a pole of order one at the origin. We complete the path of integration by taking a semi-circle of radius ε around the origin and deforming the path of integration to pass through the saddlepoint w^0 on the real axis. If we include the pole within the contour we need to evaluate the residue; if not, we can apply Cauchy's theorem directly. Both paths give the same final result and we illustrate with the path that includes the pole as in Figure 1.

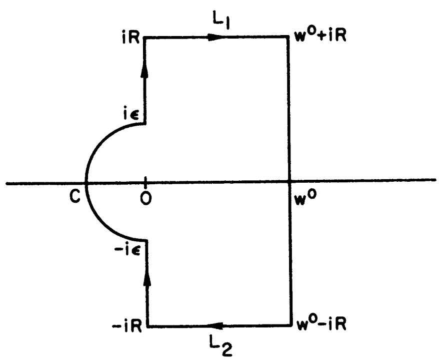

FIGURE 1.—w-plane, with w^0 a saddlepoint of the integrand and the origin a pole.

We deduce from the residue theorem that

$$\left(\int_{i\varepsilon}^{iR}+\int_{-iR}^{-i\varepsilon}\right)\frac{L(-wr,\,w)}{w}\,dw=-2\pi i\quad\text{(residue of integrand at origin)} \tag{B.3}$$

$$-\left\{\int_C+\int_{w^0+iR}^{w^0-iR}\right\}\frac{L(-wr,\,w)}{w}\,dw-\left\{\int_{L_1}+\int_{L_2}\right\}\frac{L(-wr,\,w)}{w}\,dw.$$

To evaluate the right side we note that the residue of $L(-wr,\,w)/w$ at the origin is just $L(0,0)=1$ and

$$\int_C\frac{L(-wr,\,w)}{w}\,dw=\int_{+3\pi/2}^{\pi/2}\frac{L(-\varepsilon\,e^{i\theta}r,\,\varepsilon\,e^{i\theta})}{\varepsilon\,e^{i\theta}}\,\varepsilon\,e^{i\theta}i\,d\theta.$$

Shrinking the semi-circle C to the origin we see that this integral becomes

$$i\int_{3\pi/2}^{\pi/2}d\theta=-\pi i.$$

We now find the saddlepoint approximation to the integral that passes through w^0 on the real axis. We have

$$\int_{w^0-iR}^{w^0+iR} \frac{L(-wr, w)}{w} dw = iD(w^0) \exp\left\{\frac{\mu^2}{2} \Psi(w^0)\right\} \int_{-R}^{R} \exp\left\{-\frac{\mu^2}{2} \Psi''(w^0)y^2\right\}$$
$$\cdot \exp\left[\frac{\mu^2}{2}\{-\tfrac{1}{6} \Psi^{(3)}(w^0)iy^3 - \tfrac{1}{24} \Psi^{(4)}(w^0)y^4 + \ldots\}\right]$$
$$\cdot \left[1 + \frac{D'(w^0)}{D(w^0)} iy - \frac{1}{2} \frac{D''(w^0)}{D(w^0)} y^2 + \ldots\right] dy$$

where $D(w) = C(w)/w$. When $R \to \infty$ we find, as in Section 3 of the paper, that the integral has the expansion

$$\sqrt{2\pi} D(w^0) \exp\left\{\frac{\mu^2}{2} \Psi(w^0)\right\}\left\{\frac{\mu^2}{2} \Psi''(w^0)\right\}^{-\frac{1}{2}} [1 + O(\mu^{-2})].$$

Checking the order of magnitude of $L(-wr, w)/w$ on the horizontal contours L_1 and L_2 in Figure 1, we find that the last two integrals of (B.3) tend to zero as $R \to \infty$. It now follows that

$$\lim_{\substack{\varepsilon \to \infty \\ R \to \infty}} \left(\int_{i\epsilon}^{iR} + \int_{-iR}^{-i\epsilon}\right) \frac{L(-wr, w)}{w} dw$$
$$= -2\pi i - \left\{-\pi i - \frac{\sqrt{2\pi} i D(w^0) \exp\left\{\frac{\mu^2}{2} \Psi(w^0)\right\}}{\{\mu^2/2\ \Psi''(w^0)\}^{\frac{1}{2}}}\right\}$$
$$= -\pi i + \frac{2\pi^{\frac{1}{2}} i}{\mu} \frac{D(w^0)}{\{\Psi''(w^0)\}^{\frac{1}{2}}} \exp\left\{\frac{\mu^2}{2} \Psi(w^0)\right\}.$$

From (B.1) we deduce that

$$F(r) + F(r-0) = 2 - \frac{2}{\pi^{\frac{1}{2}}\mu} \frac{D(w^0)}{\{\Psi''(w^0)\}^{\frac{1}{2}}} \exp\left\{\frac{\mu^2}{2} \Psi(w^0)\right\} [1 + O(\mu^{-2})]. \tag{B.4}$$

When the saddlepoint w^0 is on the left side of the origin we find in a similar way the formula

$$F(r) + F(r-0) = \frac{2}{\pi^{\frac{1}{2}}\mu} \frac{D(w^0)}{\{\Psi''(w^0)\}^{\frac{1}{2}}} \exp\left\{\frac{\mu^2}{2} \Psi(w^0)\right\} [1 + O(\mu^{-2})]. \tag{B.5}$$

Now consider the case where $w^0 = (r-\beta)/(1+\beta r)$. We find from (B.4) and (B.5) and the continuity of $F(r)$ that (assuming $\beta > 0$)

$$F(r) = 1 - \frac{1}{\sqrt{2\pi}\mu} \frac{(1+\beta r)^{T-K_1-1}(1+2\beta r-\beta^2)^{-(K_2-1)/2}[(1+\beta r)^2 + 2rh(r-\beta r) - h^2(r-\beta)^2]^{-(T-K)/2}}{(1+r^2)^{(K_2-2)/2}(r-\beta)}$$
$$\cdot \exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{1+r^2}\right\}[1 + O(\mu^{-2})], \qquad r > \beta,$$
$$= \frac{1}{\sqrt{2\pi}\mu} \frac{(1+\beta r)^{T-K_1-1}(1+2\beta r-\beta^2)^{-(K_2-1)/2}[(1+\beta r)^2 + 2rh(r-\beta)(1+\beta r) - h^2(r-\beta)^2]^{-(T-K)/2}}{(1+r^2)^{(K_2-2)/2}(\beta - r)}$$
$$\cdot \exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{1+r^2}\right\}[1 + O(\mu^{-2})], \qquad \frac{\beta^2-1}{2\beta} < r < \beta.$$

Differentiating the above we find that the dominant term agrees with the saddlepoint approximation to the probability density function. For values of $r < (\beta^2-1)/2\beta$ we use the saddlepoint $w^0 = -1/\beta$, as in Section 3.

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