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A Large Deviation Limit Theorem for Multivariate Distributions*

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A local limit theorem for large deviations of $o(n)^{1/2}$, where n is the sample size, is developed for multivariate statistics which are more general than standardised means, but which depend on n in much the same way. In particular, the cumulants of the statistic are of the same order in $n^{-1/2}$ as those of a standardised mean. The theory is derived under conditions which correspond to those in earlier work by Richter on limit theorems for standardised means and by Chambers on the validity of Edgeworth expansions for multivariate statistics.

1. INTRODUCTION

In recent years limit theorems for large deviations have attracted much attention on the theory of probability. A most extensive survey of this research is contained in Chapters 6–14 and Chapter 20 of Ibragimov and Linnik's excellent treatise [5]. Virtually all the results available so far seem to have been established for standardised sums of independent (and often identically distributed) random variables. This is rather unfortunate for research workers in mathematical statistics and associated areas such as econometric theory, for a major potential application of large deviation limit theory lies in approximating the tails of the finite sample distribution of statistics which are more general than standardised means, but which depend on the sample size in much the same way. In point of fact, this motivation lay behind Daniels' original work on saddlepoint approximations [2, 3], which led in turn to the systematic exploitation of this method in Richter's seminal paper [6].

Let us suppose that we are interested in the $p \times 1$ random vector Z_n whose distribution depends on the parameter n (the sample size) and whose mathematical expectation is zero. We require the higher order cumulants of $n^{1/2}Z_n$ to

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exist and be of $O(n)$ as $n \to \infty$. Then, in addition to representing a standardised mean, Z_n can represent for instance a suitable standardised second moment matrix from a multivariate distribution. In this respect, the present paper generalises Richter's multidimensional local limit theorem in [7].

In [1] Chambers dealt with multivariate statistics such as Z_n and developed asymptotic expansions of the Edgeworth type to the distributions of Z_n. Chambers also demonstrated how these expansions can be transferred to derive further expansions for the distributions of statistics which are well-behaved functions of Z_n. In deducing the validity of these expansions Chambers imposed a condition on the tails of the characteristic function of Z_n. This condition is more restrictive than Cramer's condition (C) (see, for example, [5, p. 98]) on the characteristic function of the component variables in a standardised mean. Nevertheless, it is sufficiently general to include a wide class of distributions and, in particular, Chambers applies it to central and noncentral Wishart distributions.

The aim of the present paper is to show how a limit theory for large deviations can be developed for statistics like Z_n under a Chambers-type condition on the characteristic function. This theory should then be useful in developing approximations to the tail probabilities of such distributions.

We use the term "large deviation" in this paper to refer specifically to the case of a deviation of $o(n^{1/2})$. Thus, our theory is to be distinguished from, on the one hand, the theory of large deviations of $O(n^{1/2})$ (which are referred to as very large deviations in [5]) and, on the other hand, the theory of moderate deviations which deals with deviations of $O((\log n)^{1/2})$ (see, for instance, [4]).

2. Theorem and Conditions

We impose the following conditions on the statictic Z_n:

Condition 1. The mean vector of Z_n is zero and the covariance matrix of Z_n has a positive definite limit as $n \to \infty$. All higher order cumulants of $n^{1/2}Z_n$ exist and are of $O(n)$ as $n \to \infty$.

Condition 2. There exist positive numbers A, l_n, and L_n such that in the sphere $\| z \| < A$ we have

$$l_n \leqslant \left| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{z'\zeta}\, dV(\zeta) \right| \leqslant L_n ,$$

where $\| z \| = (z^* z)^{1/2}$ in which z^* represents the complex conjugate transpose of z and $V(\zeta)$ is the distribution function of $n^{1/2}Z_n$.

CONDITION 3. The characteristic function $\phi(\theta)$ of Z_n satisfies

$$\int_{\|\theta\|>Bn^\alpha} |\phi(\theta)|\, d\theta = O(e^{-b_0 n})$$

for all $B > 0$ and for some α such that $0 < \alpha < \frac{1}{2}$ and for some $b_0 > 0$.

Under these conditions we have the result:

THEOREM. *Suppose Conditions 1–3 are satisfied and let $p_n(x)$ denote the density of Z_n. Then for $x_j > 1$ and $x_j = x_{nj} = o(n^{1/2})$ as $n \to \infty$ ($j = 1,..., p$), we have*

$$p_n(x) = \frac{1}{(2\pi)^{p/2}(\det(H_n))^{1/2}} \exp\left\{-\tfrac{1}{2}x'H_n^{-1}x + n\Psi_n\left(\frac{x}{n^{1/2}}\right)\right\}\left[1 + O\left(\frac{\|x\|}{n^{1/2}}\right)\right]$$

$$p_n(-x) = \frac{1}{(2\pi)^{p/2}(\det(H_n))^{1/2}} \exp\left\{-\tfrac{1}{2}x'H_n^{-1}x + n\Psi_n\left(\frac{-x}{n^{1/2}}\right)\right\}\left[1 + O\left(\frac{\|x\|}{n^{1/2}}\right)\right],$$

where H_n is the covariance matrix of Z_n and $\Psi_n(y)$ is a power series converging for all sufficiently small values of $\|y\|$.

Remarks. Condition 1 defines the main characteristics of the class of random vectors with which we are concerned. Note that cumulants of order $r \geqslant 2$ of Z_n are of $O(n^{-(r/2)+1})$, the same order in $1/n^{1/2}$ as for a standardised mean (when the component variables have enough cumulants).

Condition 2 implies that the characteristic function of $n^{1/2}Z_n$ is analytic in the sphere $\|z\| < A$. It is clear that in the case where $Z_n = (X_1 + \cdots + X_n)/n^{1/2}$ and the X_j ($j = 1,..., n$) are independently distributed random vectors, Condition 2 above is satisfied when (c.f. [5]) there exist positive numbers A, l, and L for which

$$l \leqslant \left| \int_{-\infty}^{\infty} e^{z'\xi}\, dV_j(\xi) \right| \leqslant L$$

in the sphere $\|z\| < A$, where $V_j(\xi)$ is the distribution function of X_j. Moreover, in this case, $l_n = l^n$ and $L_n = L^n$.

Condition 3 is a Chambers-type condition on the tails of the characteristic function of Z_n.

3. PROOF OF THE THEOREM

Since $\phi(\theta)$ is absolutely integrable (from Condition 3) the density $p_n(x)$ of Z_n exists and is given by the inversion formula

$$p_n(x) = (1/2\pi)^p \int_{R^p} e^{-ix'\theta}\phi(\theta)\, d\theta,$$

where R^p denotes p-dimensional Euclidean space. Dividing R^p into the domains

$$\Omega = \{\theta \mid |\theta_j| \leqslant \epsilon n^{1/2}, \text{ all } j\}$$

and $R^p - \Omega$, where ϵ is any positive quantity, we obtain from Condition 3

$$p_n(x) = (1/2\pi)^p \int_{\Omega} e^{-ix'\theta} \phi(\theta)\, d\theta + O(e^{-b_0 n}),$$

since for large enough n

$$\left| \int_{R^p - \Omega} e^{-ix'\theta} \phi(\theta)\, d\theta \right| < \int_{\|\theta\| > Bn^\alpha} |\phi(\theta)|\, d\theta = O(e^{-b_0 n}).$$

Let $\tau = \theta / n^{1/2}$ so that

$$p_n(x) = (n^{1/2}/2\pi)^p \int_{\Omega^*} e^{-in^{1/2}x'\tau} \phi(n^{1/2}\tau)\, d\tau + O(e^{-b_0 n}), \tag{1}$$

where

$$\Omega^* = \{\tau \mid |\tau_j| \leqslant \epsilon, \text{ all } j\}.$$

We write

$$\phi(n^{1/2}\tau) = M(in^{1/2}\tau) = \int_{R^p} e^{i\tau'\zeta}\, dV(\zeta) \tag{2}$$

and then the integrand on the right-hand side of (1) has the form

$$e^{-n^{1/2}x'u} M(n^{1/2}u) \tag{3}$$

and the paths of integration in the planes of the u_j ($j = 1,..., p$) are along the imaginary axes over the domain defined by Ω^*.

In view of (2) and Condition 2, $M(n^{1/2}u)$ has an analytic continuation to strips in the space of complex u for which $\| \operatorname{Re}(u) \| < A$. We note that

$$| M(n^{1/2}u) | > l_n \tag{4}$$

for all u in such strips and, therefore, for u which lie in

$$G = \{u \mid |u_j| \leqslant A_1 = A/2p; j = 1,..., p\}.$$

It follows from (4) that for $u \in G$ we can define $K(n^{1/2}u)$ as that branch of $\log(M(n^{1/2}u))$ for which $K(0) = 0$. Moreover, for $u \in G$, $K(n^{1/2}u)$ is an analytic function of u with the Taylor expansion about the origin given by

$$K(n^{1/2}u) = \sum_{j=2}^{\infty} (n^{j/2}/j!)[u_1(\partial/\partial u_1) + \cdots + u_p(\partial/\partial u_p)]^j K(0).$$

Using the tensor summation convention of a repeated suffix, we write this expansion as

$$(n/2)\, k_{jl}u_{jl} + (n^{3/2}/6)\, k_{jlmn}u_j u_l u_m + (n^2/24)\, k_{jlmn}u_j u_l u_m u_n$$
$$+ \cdots = n(\tfrac{1}{2}j_{jl}u_j u_l + (n^{1/2}/6)\, k_{jlm}u_j u_l u_m + (n/24)\, k_{jlmn}u_j u_l u_m u_n + \cdots), \quad (5)$$

where, for instance

$$k_{jlm} = [\partial^3 K(0)/(\partial u_j\, \partial u_l\, \partial u_m)].$$

We also note that

$$\frac{\partial K(n^{1/2}u)}{\partial u_r} = n(k_{rl}u_l + (n^{1/2}/2)\, k_{rlm}u_l u_m + (n/6)\, k_{rlmn}u_l u_m u_n + \cdots)$$
$$(r = 1,\dots, p)$$

and these power series converge uniformly for $u \in G$.

Now we can write (3) as

$$\exp\{-n^{1/2}x'u + K(n^{1/2}u)\} = \exp\{-n[y'u - (1/n)\, K(n^{1/2}u)]\},$$

where $y = x/n^{1/2}$. Thus, taking $\epsilon < A_1$, we rewrite (1) as

$$p_n(x) = (n^{1/2}/2\pi i)^p \int_{-i\epsilon}^{i\epsilon} \cdots \int_{-i\epsilon}^{i\epsilon} \exp\{-n[y'u - (1/n)\, K(n^{1/2}u)]\}\, du + O(e^{-b_0 n}) \quad (6)$$

and the paths of integration in the planes of the u_j are along the imaginary axis. We now deform the paths of integration to become the lines of steepest descent passing through the saddlepoints $u_1{}^0,\dots, u_p{}^0$ which satisfy

$$y_r = \frac{1}{n}\frac{\partial K(n^{1/2}u)}{\partial u_r} = k_{rl}u_l + \frac{n^{1/2}}{2} k_{rlm}u_l u_m + \frac{n}{6} k_{rlmn}u_l u_m u_n + \cdots$$
$$(r = 1,\dots, p) \qquad (7)$$

We take $x_j > 1$ and see that when $x_j = o(n^{1/2})$ for $j = 1,\dots, p$, we have $y = o(1)$ as $n \to \infty$. Thus, for large enough n, y will be small and the power series defined in (7) can be inverted to give power series in the y_r $(r = 1,\dots, p)$ that converge for large enough n. This follows from the fact that the matrix $H_n = [(k_{ij})]$ is positive definite for large n according to Condition 1. Thus, the positions of the saddlepoints are given by

$$u_r{}^0 = u_r{}^0(y) = a_{rj}y_j + a_{rjl}y_j y_l + a_{rjlm}y_j y_l y_m + \cdots \qquad (r = 1,\dots, p), \quad (8)$$

where

$$a_{rj} = k^{rj},$$

$$a_{rjl} = -k^{rs_0}[(n^{1/2}/2)\, k_{s_0 s_1 s_2}]\, k^{s_1 j} k^{s_2 l},$$

$$a_{rjlm} = 2k^{rs_0}[(n^{1/2}/2)\, k_{s_0 s_1 s_2}]\, k^{s_1 j} k^{s_2 s_3}[(n^{1/2}/2)\, k_{s_3 s_4 s_5}]\, k^{s_4 l} k^{s_5 m}$$

$$- k^{rs_0}[(n/6)\, k_{s_0 s_1 s_2 s_3}]\, k^{s_1 j} k^{s_2 l} k^{s_3 m}.$$

From (8) the u_r^0 ($r = 1,\ldots, p$) lie on the real axes in the complex planes of $u_1, \ldots, u_p$.

In the space of each variate u_r in the integral (6) we now consider the contour

$$L^{(r)} = L_1^{(r)} + L_2^{(r)} + L_3^{(r)} + L_4^{(r)},$$

where if $u_r^0 > 0$ we define

$$L_1^{(r)} = (i\epsilon, -i\epsilon), \qquad L_2^{(r)} = (-i\epsilon, u_r^0 - i\epsilon);$$
$$L_3^{(r)} = (u_r^0 - i\epsilon, u_r^0 + i\epsilon), \qquad L_4^{(r)} = (u_r^0 + i\epsilon, i\epsilon);$$

and if $u_r^0 < 0$ we define

$$L_1^{(r)} = (-i\epsilon, i\epsilon), \qquad L_2^{(r)} = (i\epsilon, u_r^0 + i\epsilon);$$
$$L_3^{(r)} = (u_r^0 + i\epsilon, u_r^0 - i\epsilon), \qquad L_4^{(r)} = (u_r^0 - i\epsilon, -i\epsilon).$$

We proceed to deform the paths of integration in (6) sequentially starting with u_p. We assume $u_p^0 < 0$ and the argument for the case $u_p^0 > 0$ follows a similar line.

By Cauchy's theorem we have

$$p_n(x) = -(n^{1/2}/2\pi i)^p \int_{-i\epsilon}^{i\epsilon} \cdots \int_{-i\epsilon}^{i\epsilon} \left\{ \int_{L_2^{(p)}} + \int_{L_3^{(p)}} + \int_{L_4^{(p)}} \right\}$$
$$\times \exp\{-n[y'u - (1/n)\, K(n^{1/2}u)]\}\, du_1 \cdots du_p + O(e^{-b_0 n}).$$

On the horizontal segments of the contour $L^{(p)}$ we obtain, for instance for $L_2^{(p)}$,

$$\left| (n^{1/2}/2\pi i)^p \int_{-i\epsilon}^{i\epsilon} \cdots \int_{-i\epsilon}^{i\epsilon} \int_{L_2^{(p)}} \exp\{-n(y'u) + K(n^{1/2}u)\}\, du_1 \cdots du_p \right|$$
$$\leqslant (n^{1/2}/2\pi)^p \int_{-i\epsilon}^{i\epsilon} \cdots \int_{-i\epsilon}^{i\epsilon} \int_{L_2^{(p)}} |\exp\{-ny_p \operatorname{Re}(u_p) + K(n^{1/2}u)\}|\, du_1 \cdots du_p, \quad (9)$$

where $\operatorname{Re}(u_p)$ denotes the real part of u_p. Since $u \in G$ for large enough n we can replace $K(n^{1/2}u)$ by its Taylor expansion (5). Noting the order of magnitude of the cumulants of Z_n, we have

$$K(n^{1/2}u) = n(\tfrac{1}{2}u'H_n u + O(\| u \|^3)).$$

We now introduce an orthogonal matrix C_n for which

$$C_n'H_nC_n = \operatorname{diag}(\lambda_1, \ldots, \lambda_p).$$

From Condition 1, C_n has a nonsingular limit as $n \to \infty$. We let

$$w = C_n'u = \operatorname{Re}(w) + i\operatorname{Im}(w),$$

say, and then for u_p on $L_2^{(p)}$ we have $\operatorname{Re}(w) = o(1)$ as $n \to \infty$ since $\operatorname{Re}(u_p)$ satisfies

$$u_p{}^0 \leqslant \operatorname{Re}(u_p) \leqslant 0$$

and $u_p{}^0$ from (8) has the same order as the elements of y. Transforming variables in the integral on the right side of (9) we get

$$K(n^{1/2}C_nw) = n\left(\tfrac{1}{2}\sum_{r=1}^{p} \lambda_r w_r{}^2 + O(\| w \|^3)\right)$$

and noting that

$$y_p \operatorname{Re}(u_p) = o(1)$$

we have

$$\begin{aligned}
&| \exp\{-ny_p \operatorname{Re}(u_p) + K(n^{1/2}u)\}| \\
&\quad \leqslant \left| \exp\left[n\left(\tfrac{1}{2}\sum_{r=1}^{p} \lambda_r(\operatorname{Re}(w_r)^2 - \operatorname{Im}(w_r)^2) + O(\| w \|^3) + o(1)\right)\right]\right| \\
&\quad \leqslant \exp\left\{-(n/4)\sum_{r=1}^{p} \lambda_r \operatorname{Im}(w_r)^2\right\} \qquad (10)
\end{aligned}$$

for large enough n and small enough ϵ since $\operatorname{Im}(w_r) = O(\epsilon)$. It now follows from (10) that

$$\begin{aligned}
&(n^{1/2}/2\pi)^p \int_{-i\epsilon}^{i\epsilon} \cdots \int_{-i\epsilon}^{i\epsilon} \int_{L_2^{(p)}} | \exp\{-ny_p \operatorname{Re}(u_p) + K(n^{1/2}u)\}| \, du_1 \cdots du_p \\
&\quad = O(n^{p/2}e^{-b_1 n}) = O(e^{-b_2 n})
\end{aligned}$$

as $n \to \infty$, where b_1 is some positive constant and $0 < b_2 < b_1$.

We obtain a similar expression for the integral over the contour $L_4^{(p)}$. Thus

$$p_n(x) = (n^{1/2}/2\pi i)^p \int_{-i\epsilon}^{i\epsilon} \cdots \int_{-i\epsilon}^{i\epsilon} \int_{u_p{}^0 - i\epsilon}^{u_p{}^0 + i\epsilon} \exp\{-n[y'u - (1/n)\, K(n^{1/2}u)]\}\, du_1 \cdots du_p$$

$$+ O(e^{-b_3 n})$$

where $b_3 = \min(b_0, b_2)$.

Deforming the path of integration in the planes of $u_{p-1}, u_{p-2}, \ldots, u_1$ we find that in each case the integral over the horizontal segments $L_2^{(r)}$ and $L_4^{(r)}$ can be neglected, leaving us with

$$p_n(x) = (n^{1/2}/2\pi i)^p \int_{u_1{}^0 - i\epsilon}^{u_1{}^0 + i\epsilon} \cdots \int_{u_p{}^0 - i\epsilon}^{u_p{}^0 + i\epsilon} \exp\{-n[y'u - (1/n)\, K(n^{1/2}u)]\}\, du_1 \cdots du_p$$

$$+ O(e^{-bn}) \tag{11}$$

for some positive constant b.

Along the paths of integration in (11) we have

$$u_r = u_r{}^0 + it_r, \qquad r = 1, \ldots, p, \tag{12}$$

where

$$-\epsilon \leqslant t_r \leqslant \epsilon. \tag{13}$$

For large enough n, small enough ϵ and u satisfying (12) and (13) we can expand

$$(1/n)\, K(n^{1/2}u) - y'u$$

in a Taylor series about u^0. We have

$$(1/n)\, K(n^{1/2}u) - y'u = (1/n)\, K(n^{1/2}u^0) - y'u^0$$

$$+ (1/n) \sum_{j=2}^{\infty} (1/j!)[it_1(\partial/\partial u_1) + \cdots + it_p(\partial/\partial u_p)]^j\, K(n^{1/2}u^0).$$

Now

$$\frac{1}{n} K(n^{1/2}u^0) - y'u^0 = \frac{1}{n} K(n^{1/2}u^0) - \frac{1}{n}\left(\frac{\partial K(n^{1/2}u^0)}{\partial u}\right)' u^0$$

$$= \frac{1}{n} \sum_{j=2}^{\infty} \left(\frac{j-1}{j!}\right) n^{j/2} \left(u_1{}^0 \frac{\partial}{\partial u_1} + \cdots + u_p{}^0 \frac{\partial}{\partial u_p}\right)^j K(0), \tag{14}$$

and from (8) and (14) we obtain after some algebra

$$(1/n)\, K(n^{1/2}u^0) - yu^0 = -\tfrac{1}{2} y' H_n^{-1} y + \Psi_n(y), \tag{15}$$

where

$$\begin{aligned}\Psi_n(y) = {} & (n^{1/2}/6)\, k_{s_0 s_1 s_2} k^{s_0 a} k^{s_1 b} k^{s_2 c} y_a y_b y_c + (n/24)(k_{s_0 s_1 s_2 s_3} k^{s_0 a} k^{s_1 b} k^{s_2 c} k^{s_3 d} \\ & - 3 k_{r_0 r_1 r_2} k_{r_3 r_4 r_5} k^{r_2 r_3} k^{r_0 a} k^{r_1 b} k^{r_4 c} k^{r_5 d})\, y_a y_b y_c y_d + \cdots,\end{aligned} \tag{16}$$

and k^{ij} represents the (i, j)th element of H_n^{-1}. The coefficients in the powers series $\Psi(y)$ are completely determined by the cumulants of Z_n. These coefficients are of $O(1)$ as $n \to \infty$ and the series converges for sufficiently small y.

Returning to (11) we now have

$$\begin{aligned}p_n(x) = {} & (n^{1/2}/2\pi)^p \int_{-\epsilon}^{\epsilon} \cdots \int_{-\epsilon}^{\epsilon} \exp\{n[(1/n)\, K(n^{1/2}u^0) - y'u^0]\} \\ & \times \exp\left\{n\left[(1/n) \sum_{j=2}^{\infty} (1/j!)[it_1(\partial/\partial u_1) + \cdots + it_p(\partial/\partial u_p)]^j\, K(n^{1/2}u^0)\right]\right\} \\ & \times dt_1 \cdots dt_p + O(e^{-bn}).\end{aligned} \tag{17}$$

Moreover

$$\begin{aligned}(1/n) & \sum_{j=2}^{\infty} (1/j!)[it_1(\partial/\partial u_1) + \cdots + it_p(\partial/\partial u_p)]^j\, K(n^{1/2}u^0) \\ & = -(1/2n)[\partial^2 K(n^{1/2}u^0)/\partial u_a\, \partial u_b]\, t_a t_b + O(\|t\|^3)\end{aligned} \tag{18}$$

and

$$(1/n)[\partial^2 K(n^{1/2}u^0)/\partial u_a\, \partial u_b] = k_{ab} + o(1),$$

so that for large enough n

$$(1/n)[\partial^2 K(n^{1/2}u^0)/\partial u_a\, \partial u_b]\, t_a t_b > \tfrac{1}{2} k_{ab} t_a t_b = \tfrac{1}{2} t' H_n t. \tag{19}$$

Following the line of argument in Ibragimov and Linnik [4, p. 165] we separate the domain of integration in (17) into the two regions:

$$0 \leqslant |t_r| \leqslant n^{-1/2}(\log n)^2, \qquad r = 1, \ldots, p \tag{20}$$

and

$$n^{-1/2}(\log n)^2 \leqslant |t_r| \leqslant \epsilon, \qquad r = 1, \ldots, p. \tag{21}$$

For t in the region defined by (21) we see from (18) and (19) that

$$\operatorname{Re}\left\{\sum_{j=2}^{\infty}(1/j!)[it_1(\partial/\partial u_1)+\cdots+it_p(\partial/\partial u_p)]^j K(n^{1/2}u^0)\right\} < -(n/8)\,t'H_n t$$

for large enough n and small enough ϵ. Moreover, from the inequality

$$t'H_n t \geqslant \bar{\lambda}_n t't,$$

where $\bar{\lambda}_n$ represents the smallest eigenvalue of H_n, we deduce that if $\bar{\lambda}$ is the limit of $\bar{\lambda}_n$ as $n \to \infty$ then, for large enough n,

$$t'H_n t > (\bar{\lambda}-\eta)\,t't,$$

where η is a small positive quantity for which $0 < \eta < \bar{\lambda}$ and we note that $\bar{\lambda}$ is strictly positive by Condition 1. Contributions to the integral (17) from regions for which any argument t_r is restricted to (21) can be neglected as of order

$$O(n^{p/2}\exp\{K(n^{1/2}u^0)-ny'u^0\}\exp\{-b_4(\log n)^4\})$$

for some positive constant b_4. To see this we can, for instance, consider the region

$$\Gamma = \{t \mid n^{-1/2}(\log n)^2 \leqslant t_p \leqslant \epsilon;\ -\epsilon \leqslant t_j \leqslant \epsilon, j = 1,\ldots, p-1\},$$

and then

$$\begin{aligned}\sup_{t\in\Gamma}\{\exp(-(n/8)\,t'H_n t)\} &\leqslant \exp\{-(n/8)(\bar{\lambda}-\eta)\inf_{t\in\Gamma} t't\}\\ &= \exp\{-\tfrac{1}{8}(\bar{\lambda}-\eta)(\log n)^4\},\end{aligned}$$

so that

$$\int_\Gamma \exp(-(n/8)\,t'H_n t)\,dt = O(\exp\{-\tfrac{1}{8}(\bar{\lambda}-\eta)(\log n)^4\})$$

as $n \to \infty$, since Γ has volume at most of $O(1)$.

Thus (17) becomes

$$\begin{aligned}p_n(x) = {}&\left(\frac{n^{1/2}}{2\pi}\right)^p \exp\{K(n^{1/2}u^0)-ny'u^0\}\\ &\times \int_{-n^{-1/2}(\log n)^2}^{n^{-1/2}(\log n)^2}\cdots\int_{-n^{-1/2}(\log n)^2}^{n^{-1/2}(\log n)^2} \exp\left\{\sum_{j=2}^{\infty}\frac{1}{j!}\left(it_1\frac{\partial}{\partial u_1}+\cdots+it_p\frac{\partial}{\partial u_p}\right)^j\right.\\ &\left.\times K(n^{1/2}u^0)\right\} dt_1\cdots dt_p\\ &+ O(n^{p/2}\exp\{K(n^{1/2}u^0)-ny'u^0\}\exp(-b_4(\log n)^4))\\ &+ O(e^{-\delta n}).\end{aligned} \tag{22}$$

For t in the region defined by (20) we have for large enough n

$$\exp\left\{\sum_{j=2}^{\infty}\frac{1}{j!}\left(it_1\frac{\partial}{\partial u_1}+\cdots+it_p\frac{\partial}{\partial u_p}\right)^j K(n^{1/2}u^0)\right\}$$
$$=-\frac{1}{2}\frac{\partial^2 K(n^{1/2}u^0)}{\partial u_a\,\partial u_b}t_a t_b$$
$$+\frac{1}{6}\frac{\partial^3 K(n^{1/2}u^0)}{\partial u_a\,\partial u_b\,\partial u_c}(it_a)(it_b)(it_c)+O(n^{-1}(\log n)^8).$$

Then we get for the integral in (22)

$$\int_{-n^{-1/2}(\log n)^2}^{n^{-1/2}(\log n)^2}\cdots\int_{-n^{-1/2}(\log n)^2}^{n^{-1/2}(\log n)^2}\exp\left\{-\frac{1}{2}\frac{\partial^2 K(n^{1/2}u^0)}{\partial u_a\,\partial u_b}t_a t_b\right\}$$
$$\times\left[1+\frac{1}{6}\frac{\partial^3 K(n^{1/2}u^0)}{\partial u_a\,\partial u_b\,\partial u_c}(it_a)(it_b)(it_c)+O(n^{-1}(\log n)^8)\right]dt_1\cdots dt_p$$
$$=\int_{-n^{-1/2}(\log n)^2}^{n^{-1/2}(\log n)^2}\cdots\int_{-n^{-1/2}(\log n)^2}^{n^{-1/2}(\log n)^2}\exp\left\{-\frac{1}{2}\frac{\partial^2 K(n^{1/2}u^0)}{\partial u_a\,\partial u_b}t_a t_b\right\}$$
$$\times[1+O(n^{-1}(\log n)^8)]\,dt_1\cdots dt_p$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\frac{\partial^2 K(n^{1/2}u^0)}{\partial u_a\,\partial u_b}t_a t_b\right\}dt_1\cdots dt_p[1+O(n^{-1}(\log n)^8)]$$
$$=(2\pi)^{p/2}\left\{\det\left(\frac{\partial^2 K(n^{1/2}u^0)}{\partial u\,\partial u'}\right)\right\}^{-1/2}[1+O(n^{-1}(\log n)^8)].$$

Now

$$\det[\partial^2 K(n^{1/2}u^0)/(\partial u\,\partial u')]=n^p(\det(H_n)+O(u^0)),$$

so that since the components of u^0 have the same order as the components of y and these are by definition of $O(x/n^{1/2})$, we have

$$\left\{\det\left(\frac{\partial^2 K(n^{1/2}u^0)}{\partial u\,\partial u'}\right)\right\}^{-1/2}=n^{-p/2}(\det(H_n))^{-1/2}(1+O(\|x\|/n^{1/2}))$$
$$=\frac{n^{-p/2}}{(\det(H_n))^{1/2}}(1+O(\|x\|\,n^{1/2})).$$

It follows that

$$
\begin{aligned}
p_n(x) = {} & \frac{1}{(2\pi)^{p/2}(\det(H_n))^{1/2}} \exp\{K(n^{1/2}u^0) - ny'u^0\} \\
& \times [1 + O(\| x \|/n^{1/2})][1 + O(n^{-1}(\log n)^8)] \\
& + O(n^{p/2} \exp\{K(n^{1/2}u^0) - nyu^0\} \exp(-b_4(\log n)^4)) \\
& + O(e^{-bn}).
\end{aligned} \tag{23}
$$

Taking the second term on the right side of (23) we have

$$n^{p/2} \exp(-b_4(\log n)^4) = \exp\{(\log n)[(p/2) - b_4(\log n)^3]\} = O(n^{-k}) \tag{24}$$

for any $k > 0$ as $n \to \infty$. From (15) we also have

$$\exp\{K(n^{1/2}u^0) - ny'u^0\} = \exp\{-(n/2)\, y'H_n^{-1}y + n\Psi_n(y)\} = O(\exp(-b_5 n/\rho(n)), \tag{25}$$

where b_5 is some positive constant and $\rho(n) \to \infty$ as $n \to \infty$. And taking the third term on the right-hand side of (23) we see that for any $k > 0$

$$e^{-bn} = o(\exp\{K(n^{1/2}u^0) - ny'u^0\}\, n^{-k}). \tag{26}$$

Using (24), (25), and (26) in (23) we find that

$$
\begin{aligned}
p_n(x) = {} & \frac{1}{(2\pi)^{p/2}(\det(H_n))^{1/2}} \exp\{-\tfrac{1}{2}x'H_n^{-1}x + n\Psi_n(x/n^{1/2})\} \\
& \times [1 + O(\| x \|/n^{1/2})].
\end{aligned}
$$

Replacing x by $-x$ we obtain the corresponding expression for $p_n(-x)$. This proves the theorem.

Acknowledgments

This article draws much of its methodology from the early paper of Richter [6] and the treatment of large deviations in Ibragimov and Linnik [5]. To these writers I wish to express my thanks.

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