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APPROXIMATIONS TO SOME FINITE SAMPLE DISTRIBUTIONS ASSOCIATED WITH A FIRST-ORDER STOCHASTIC DIFFERENCE EQUATION

BY P. C. B. PHILLIPS[1]

Edgeworth series expansions are obtained of the finite sample distributions of the least squares estimator and the associated t ratio test statistic in the context of a first-order noncircular stochastic difference equation. General formulae are given for these expansions up to $0(T^{-1})$ where T is the sample size and explicit representations of these in terms of the true parameters are derived up to $O(T^{-\frac{1}{2}})$. Some numerical comparisons of the approximations and the exact distributions are made in the case of the least squares estimator.

1. INTRODUCTION

IN A RECENT ARTICLE, Basmann, et al. [**2**] have studied the finite sample distribution of least squares estimators and associated test statistics in the context of an incomplete system of stochastic difference equations. Their general conclusion, which is supported by the evidence of a sampling experiment based on 1,000 replications, is that the asymptotic normal distribution theory does not seem to provide a good guide to the finite sample distributions (given a sample size of 20) as far as these can be determined by experimental evidence.

The present paper deals with a similar subject but differs from [**2**] in two respects: In the first place, approximations to the relevant finite sample distributions are constructed on the basis of the Edgeworth series expansion rather than with experimental data. Secondly, our analysis is confined to the first-order noncircular stochastic difference equation with no exogenous variables. Naturally, this choice of model reduces the scope of the present paper. However, the results of Phillips [**6**] and Sargan [**8**] indicate that asymptotic expansions for the finite sample distributions of econometric estimators can, in principle, be obtained in much more complicated models with lagged endogenous variables and disturbances which need not be normally distributed. But, in view of the great complexity of these expansions, even in relatively small time series models, it seems worthwhile to consider how satisfactory the approximations derived from the Edgeworth expansion prove to be in the very simple case selected for study here. This should give us some guide as to whether or not the approach may be useful in more complicated time series models.

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2. THE SAMPLING DISTRIBUTION OF THE LEAST SQUARES ESTIMATOR

The model we will use is the noncircular stochastic difference equation

$$y_t = \alpha y_{t-1} + u_t \qquad (t = \ldots, -1, 0, 1, \ldots; |\alpha| < 1)$$

where the u_t are independent and identically distributed $n(0, \sigma^2)$. The least squares estimator of α is given by $\hat{\alpha} = y'C_1y/y'C_2y$ where $y' = (y_0, \ldots, y_T)$,

$$C_1 = \begin{pmatrix} 0 & \frac{1}{2} & \ldots & 0 & 0 \\ \frac{1}{2} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{1}{2} & 0 \end{pmatrix} \text{and} \quad C_2 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Introducing $\mu_i = E(y'C_iy)$ we can write

$$\hat{\alpha} - \alpha = \frac{(y'C_1y - \mu_1) - \alpha(y'C_2y - \mu_2)}{(y'C_2y - \mu_2) + \mu_2}$$

since $\mu_1 - \alpha\mu_2 = 0$. Defining $q_i = (y'C_iy - \mu_i)/T$, we can now express the error in the estimator $\hat{\alpha}$ as

$$\hat{\alpha} - \alpha = e_T(q) = \frac{q_1 - \alpha q_2}{q_2 + \mu_2/T}$$

where $q' = (q_1, q_2)$ and $e_T(q)$ satisfies the conditions of the theorem in [**6**]. Thus, $P(\sqrt{T}(\hat{\alpha} - \alpha) \leq x) = P(\sqrt{T}e_T(q) \leq x)$ can be developed in an Edgeworth expansion. A convenient algorithm for finding the terms in this expansion is based on an expansion of the characteristic function (c.f. Anderson and Sawa [**1**]) and this approach is used in what follows.

Writing $\hat{\alpha} = A/B$, we know that the joint characteristic function of (A, B) is

$$\Phi(t_1, t_2) = [\det\{I - 2i(t_1C_1 + t_2C_2)\Sigma\}]^{-\frac{1}{2}}$$

where Σ is the covariance matrix of y with (i, j)th element given by $\alpha^{|i-j|}\sigma^2/(1-\alpha^2)$. We now have

$$P(\sqrt{T}(\hat{\alpha} - \alpha) \leqslant x) = P(A - rB \leqslant 0)$$
$$= P(Q \leqslant 0),$$

where $r = \alpha + x/\sqrt{T}$ and $Q = A - rB$. The characteristic function of Q is therefore

$$\psi(t) = \Phi(t, -tr)$$
$$= [\det\{I - 2it(C_1 - rC_2)\Sigma\}]^{-\frac{1}{2}}$$
$$= \prod_{j=1}^{T+1} (1 - 2it\delta_j)^{-\frac{1}{2}},$$

where $\delta_1, \ldots, \delta_{T+1}$ are the eigenvalues of $(C_1 - rC_2)\Sigma$. The second characteristic (or cumulant generating function) of Q is

$$\begin{aligned}\psi_1(t) &= \log\{\psi(t)\}\\ &= -\tfrac{1}{2}\sum_{j=1}^{T+1}\log(1-2it\delta_j)\\ &= \tfrac{1}{2}\sum_{j=1}^{T+1}\sum_{s=1}^{\infty}\frac{1}{s}(2it\delta_j)^s\\ &= \sum_{s=1}^{\infty}\frac{2^{s-1}}{s}(it)^s\sum_{j=1}^{T+1}\delta_j^s,\end{aligned}$$

so that the sth cumulant of Q is

$$k_s = (s-1)!2^{s-1}\sum_{j=1}^{T+1}\delta_j^s = (s-1)!2^{s-1}\operatorname{tr}((C_1-rC_2)\Sigma)^s.$$

Since C_1 and Σ are Toeplitz matrices and C_2 asymptotically Toeplitz, we note that

$$\frac{1}{T+1}\sum_{j=1}^{T+1}\delta_j^s = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{2\pi f(\lambda)(\cos\lambda - r)\}^s\,d\lambda + O(T^{-1})$$

(c.f. Hannan [**3**, p. 354]) where $f(\lambda)$ is the spectral density of y_t. Thus $k_s = O(T)$ and

$$k_s' = \frac{k_s}{(k_2)^{s/2}} = O(T^{-s/2+1}).$$

Moreover, from the above formulae we obtain

$$\begin{aligned}k_s &= (s-1)!2^{s-1}\left(\frac{T}{2\pi}\right)\int_{-\pi}^{\pi}\{2\pi f(\lambda)(\cos\lambda - r)\}^s\,d\lambda + O(1)\\ &= (s-1)!2^{s-1}\left(\frac{T}{2\pi}\right)\sum_{j=0}^{s}\binom{s}{j}(-1)^{s-j}r^{s-j}\int_{-\pi}^{\pi}(2\pi f(\lambda))^s\\ &\quad\cdot(\cos\lambda)^j\,d\lambda + O(1).\end{aligned}$$

We now define $Q' = (Q-k_1)/\sqrt{k_2}$ so that

$$P(\sqrt{T}(\hat{\alpha}-\alpha)\leqslant x) = P(Q'\leqslant -k_1/\sqrt{k_2}).$$

The characteristic function of Q' is

$$\begin{aligned}\exp\{\psi_1(t/\sqrt{k_2}) - ik_1t/\sqrt{k_2}\} &= e^{-t^2/2}\exp\left\{\sum_{s=3}^{\infty}\frac{k_s'}{s!}(it)^s\right\}\\ &= e^{-t^2/2}\left\{1+\frac{k_3'}{3!}(it)^3+\frac{k_4'}{4!}(it)^4+\frac{10}{6!}(k_3')^2(it)^6\right\}\\ &\quad + O(T^{-3/2})\end{aligned}$$

and inverting we obtain the distribution function

$$P(Q'\leq w)=\int_{-\infty}^{w} i(z)\,dz+\frac{k_3'}{3!}\int_{-\infty}^{w} H_3(z)i(z)\,dz+\frac{k_4'}{4!}\int_{-\infty}^{w} H_4(z)i(z)\,dz$$
$$+\frac{10}{6!}(k_3')^2\int_{-\infty}^{w} H_6(z)i(z)\,dz+O(T^{-3/2})$$

where $H_r(z)$ is the Hermite polynomial of degree r and $i(z)=(1/2\pi)^{\frac{1}{2}}\exp(-z^2/2)$.

Thus, we have the following series expansion of the distribution function of $\sqrt{T}(\hat{\alpha}-\alpha)$:

(1) $$P(\sqrt{T}(\hat{\alpha}-\alpha)\leq x)=I\left(\frac{-k_1}{\sqrt{k_2}}\right)-\frac{k_3'}{3!}I^{(3)}\left(\frac{-k_1}{\sqrt{k_2}}\right)$$
$$+\frac{k_4'}{4!}I^{(4)}\left(\frac{-k_1}{\sqrt{k_2}}\right)+\frac{10(k_3')^2}{6!}I^{(6)}\left(\frac{-k_1}{\sqrt{k_2}}\right)+O(T^{-3/2})$$

where $I(z)=\int_{-\infty}^{z}i(t)\,dt$. But k_s is a function of $r=x/\sqrt{T}+\alpha$ for all s, so we write $k_s=k_s(r)$ and consider the Taylor development of $k_1/\sqrt{k_2}$ about the value at $r=\alpha$. We have

(2) $$k_1'=\frac{k_1}{\sqrt{k_2}}=\frac{k_1(\alpha)}{\sqrt{k_2(\alpha)}}+\frac{\partial k_1'(\alpha)}{\partial r}\frac{x}{\sqrt{T}}+\frac{1}{2}\frac{\partial^2 k_1'(\alpha)}{\partial r^2}\frac{x^2}{T}+\frac{1}{6}\frac{\partial^3 k_1'(\alpha)}{\partial r^3}\frac{x^3}{T^{3/2}}+R$$

where $R=T^{-2}(\partial^4 k_1'(r^*)/\partial r^4)x^4/24$ and r^* lies between r and α. Now

$$k_1=\frac{-xT\sigma^2}{\sqrt{T}(1-\alpha^2)}=\frac{-(r-\alpha)T\sigma^2}{1-\alpha^2}$$

so that $k_1(\alpha)=0$. Thus, we can write (2) as

(3) $$k_1'=l_1(\alpha)\frac{x}{\sqrt{T}}+l_2(\alpha)\frac{x^2}{T}+l_3(\alpha)\frac{x^3}{T^{3/2}}+R.$$

After some manipulation we obtain

$$\operatorname{tr}(C_1\Sigma)^2=\frac{1}{4}\left(\frac{\sigma^2}{1-\alpha^2}\right)^2\{2T+2(5T-4)\alpha^2$$
$$+8(T-2)\alpha^4+8(T-3)\alpha^6+\ldots+8\alpha^{2(T-1)}\},$$
$$\operatorname{tr}(C_2\Sigma)^2=\left(\frac{\sigma^2}{1-\alpha^2}\right)^2\{(T)+2(T-1)\alpha^2+2(T-2)\alpha^4+\ldots+2\alpha^{2(T-1)}\},$$
$$\operatorname{tr}(C_1\Sigma C_2\Sigma)=\frac{1}{2}\left(\frac{\sigma^2}{1-\alpha^2}\right)^2$$
$$\cdot\{2(2T-1)+2(2T-3)\alpha^3+2(2T-5)\alpha^5+\ldots+2\alpha^{2T-1}\},$$

and, therefore,

(4) $$k_2(\alpha)=2\operatorname{tr}((C_1-\alpha C_2)\Sigma)^2=\frac{T\sigma^4}{1-\alpha^2}$$

so that

$$l_1(\alpha)=\frac{\partial k_1'(\alpha)}{\partial r}=\frac{\partial k_1(\alpha)/\partial r}{\sqrt{k_2(\alpha)}}=-\frac{T\sigma^2/(1-\alpha^2)}{\{2\operatorname{tr}((C_1-\alpha C_2)\Sigma)^2\}^{\frac{1}{2}}}=-\left(\frac{T}{1-\alpha^2}\right)^{\frac{1}{2}}.$$

After some further algebra which is reported in full in [7] we find that

$$l_2(\alpha)=-\frac{2\sqrt{T}\alpha}{(1-\alpha^2)^{3/2}}+O(T^{-\frac{1}{2}})$$

and

$$l_3(\alpha)=\frac{\sqrt{T}(1+\alpha^2)}{(1-\alpha^2)^{5/2}}-\frac{6\sqrt{T}\alpha^2}{(1-\alpha^2)^{5/2}}+O(T^{-\frac{1}{2}}).$$

Writing (3) in the form

$$k_1'=-\frac{x}{\sqrt{1-\alpha^2}}+l_2(\alpha)\frac{x^2}{T}+l_3(\alpha)\frac{x^3}{T^{3/2}}+O(T^{-3/2}),$$

we can now return to (1) and consider a Taylor development of $I(-k_1')$, $I^{(3)}(-k_1')$, $I^{(4)}(-k_1')$, and $I^{(6)}(-k_1')$ about the value $x/\sqrt{1-\alpha^2}$. We obtain

$$\begin{aligned}
I(-k_1')&=I\left(\frac{x}{\sqrt{1-\alpha^2}}\right)+i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left\{-l_2(\alpha)\frac{x^2}{T}-l_3(\alpha)\frac{x^3}{T^{3/2}}\right\}\\
&\quad+\frac{1}{2}i'\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left(-l_2(\alpha)\frac{x^2}{T}\right)^2+O(T^{-3/2})\\
&=I\left(\frac{x}{\sqrt{1-\alpha^2}}\right)+i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\\
&\quad\cdot\left\{-l_2(\alpha)\frac{x^2}{T}-l_3(\alpha)\frac{x^3}{T^{3/2}}-\frac{1}{2}\frac{x}{\sqrt{1-\alpha^2}}(l_2(\alpha))^2\frac{x^4}{T^2}\right\}+O(T^{-3/2}),
\end{aligned}$$

$$\begin{aligned}
I^{(3)}(-k_1')&=I^{(3)}\left(\frac{x}{\sqrt{1-\alpha^2}}\right)+I^{(4)}\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left(-l_2(\alpha)\frac{x^2}{T}\right)+O(T^{-1})\\
&=i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left\{\frac{x^2}{1-\alpha^2}-1\right\}\\
&\quad-i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left\{\left(\frac{x}{\sqrt{1-\alpha^2}}\right)^3-3\frac{x}{\sqrt{1-\alpha^2}}\right\}\left(\frac{-l_2(\alpha)x^2}{T}\right)+O(T^{-1})\\
&=i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left\{\frac{x^2}{1-\alpha^2}-1+\frac{x^5}{(1-\alpha^2)^{3/2}}\frac{l_2(\alpha)}{T}-\frac{3x^3}{\sqrt{1-\alpha^2}}\frac{l_2(\alpha)}{T}\right\}\\
&\quad+O(T^{-1}),
\end{aligned}$$

$$\begin{aligned}
I^{(4)}(-k_1')&=I^{(4)}\left(\frac{x}{\sqrt{1-\alpha^2}}\right)+O(T^{-\frac{1}{2}})\\
&=-i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left\{\frac{x^3}{(1-\alpha^2)^{3/2}}-3\frac{x}{\sqrt{1-\alpha^2}}\right\}+O(T^{-\frac{1}{2}}),
\end{aligned}$$

and

$$I^{(6)}(-k_1') = I^{(6)}\left(\frac{x}{\sqrt{1-\alpha^2}}\right) + O(T^{-\frac{1}{2}})$$

$$= -i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left\{\frac{x^5}{(1-\alpha^2)^{5/2}} - 10\frac{x^3}{(1-\alpha^2)^{3/2}} + 15\frac{x}{\sqrt{1-\alpha^2}}\right\}$$
$$+ O(T^{-\frac{1}{2}}).$$

Hence, collecting terms in (1) we have

$$P(\sqrt{T}(\hat{\alpha}-\alpha) \leqslant x) = I\left(\frac{x}{\sqrt{1-\alpha^2}}\right) + i\left(\frac{x}{\sqrt{1-\alpha^2}}\right) \tag{5}$$
$$\cdot\left\{-l_2(\alpha)\frac{x^2}{T} - l_3(\alpha)\frac{x^3}{T^{3/2}} - \frac{1}{2}\frac{(l_2(\alpha))^2 x^5}{\sqrt{1-\alpha^2}T^2}\right\}$$
$$-\frac{k_3'}{3!}i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)$$
$$\cdot\left\{\frac{x^2}{1-\alpha^2} - 1 + \frac{l_2(\alpha)}{(1-\alpha^2)^{3/2}}\frac{x^5}{T} - \frac{3l_2(\alpha)}{\sqrt{1-\alpha^2}}\frac{x^3}{T}\right\}$$
$$+\frac{k_4'}{4!}i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left\{\frac{3x}{\sqrt{1-\alpha^2}} - \frac{x^3}{(1-\alpha^2)^{3/2}}\right\}$$
$$+\frac{10(k_3')^2}{6!}i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)$$
$$\cdot\left\{\frac{-x^5}{(1-\alpha^2)^{5/2}} + \frac{10x^3}{(1-\alpha^2)^{3/2}} - \frac{15x}{\sqrt{1-\alpha^2}}\right\}$$
$$+ O(T^{-3/2}).$$

We now expand k_3' and k_4' about the value $r=\alpha$. We have

$$k_3' = k_3'(\alpha) + \frac{\partial k_3'(\alpha)}{\partial r}\frac{x}{\sqrt{T}} + O(T^{-3/2}) = k_3'(\alpha) + h(\alpha)\frac{x}{\sqrt{T}} + O(T^{-3/2}), \tag{6}$$

say, and

$$k_4' = k_4'(\alpha) + O(T^{-3/2}). \tag{7}$$

Since $k_3 = 8\ \mathrm{tr}\ ((C_1 - rC_2))^3$ we find that

$$h(\alpha) = \frac{\partial k_3'(\alpha)}{\partial r} = -\frac{24\ \mathrm{tr}\ (C_2\Sigma((C_1-\alpha C_2)\Sigma)^2)}{\{2\ \mathrm{tr}\ ((C_1-\alpha C_2)\Sigma)^2\}^{3/2}}$$
$$+\frac{48\ \mathrm{tr}\ ((C_1-\alpha C_2)\Sigma)^3\ \mathrm{tr}\ (C_2\Sigma(C_1\Sigma - \alpha C_2\Sigma))}{\{2\ \mathrm{tr}\ ((C_1-\alpha C_2)\Sigma)^2\}^{5/2}}.$$

Using (6) and (7) in (5) we can now write

$$P(\sqrt{T}(\hat{\alpha}-\alpha)\leqslant x)=I\left(\frac{x}{\sqrt{1-\alpha^2}}\right) \tag{8}$$
$$+i\left(\frac{x}{\sqrt{1-\alpha^2}}\right)\left\{a_0+a_1\left(\frac{x}{\sqrt{1-\alpha^2}}\right)+a_2\left(\frac{x}{\sqrt{1-\alpha^2}}\right)^2\right.$$
$$\left.+a_3\left(\frac{x}{\sqrt{1-\alpha^2}}\right)^3+a_5\left(\frac{x}{\sqrt{1-\alpha^2}}\right)^5\right\}+O(T^{-3/2})$$

where

$$a_o=\tfrac{1}{6}k_3'(\alpha),$$

$$a_1=\frac{3}{4!}k_4'(\alpha)-\frac{150}{6!}(k_3'(\alpha))^2+\frac{1}{6\sqrt{T}}h(\alpha)\sqrt{1-\alpha^2},$$

$$a_2=-\frac{l_2(\alpha)(1-\alpha^2)}{T}-\frac{k_3'(\alpha)}{6}$$
$$=\left(\frac{2\alpha}{\sqrt{1-\alpha^2}}\right)\frac{1}{\sqrt{T}}-\frac{1}{6}k_3'(\alpha)+O(T^{-3/2}),$$

$$a_3=-\frac{l_3(\alpha)(1-\alpha^2)^{3/2}}{T^{3/2}}+\frac{3k_3'(\alpha)l_2(\alpha)(1-\alpha^2)}{3!\,T}$$
$$-\frac{h(\alpha)(1-\alpha^2)^{\frac{1}{2}}}{3!\,T}-\frac{k_4'(\alpha)}{4!}+\frac{100(k_3'(\alpha))^2}{6!}$$
$$=-\frac{1}{T}\left(\frac{1+\alpha^2}{1-\alpha^2}\right)+\frac{6\alpha^2}{(1-\alpha^2)T}-\frac{k_3'(\alpha)}{\sqrt{T}}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right)$$
$$-\frac{h(\alpha)\sqrt{1-\alpha^2}}{6\sqrt{T}}-\frac{k_4'(\alpha)}{4!}+\frac{100(k_3'(\alpha))^2}{6!},$$

and

$$a_5=-\frac{2}{T}\left(\frac{\alpha^2}{1-\alpha^2}\right)+\frac{k_3'(\alpha)}{3\sqrt{T}}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right)-\frac{10(k_3'(\alpha))^2}{6!}.$$

Setting $w=x/\sqrt{1-\alpha^2}$ we have the following approximation to the distribution function $P(\sqrt{T}(\hat{\alpha}-\alpha)/\sqrt{1-\alpha^2}\leqslant w)$ up to $O(T^{-1})$:

$$I(w)+i(w)(a_0+a_1w+a_2w^2+a_3w^3+a_5w^5). \tag{9}$$

An alternative representation of this approximation up to $O(T^{-1})$ is given by

$$I(w+b_0+b_1w+b_2w^2+b_3w^3)$$

where the coefficients are

$$b_o = \frac{k_3'(\alpha)}{6},$$

$$b_1 = \frac{3}{4!}k_4'(\alpha) - \frac{140}{6!}(k_3'(\alpha))^2 + \frac{1}{6\sqrt{T}}h(\alpha)\sqrt{1-\alpha^2},$$

$$b_2 = \left(\frac{2\alpha}{\sqrt{1-\alpha^2}}\right)\frac{1}{\sqrt{T}} - \frac{1}{6}k_3'(\alpha),$$

and

$$b_3 = -\frac{1}{T}\left(\frac{1+\alpha^2}{1-\alpha^2}\right) + \frac{6\alpha^2}{(1-\alpha^2)T}$$

$$-\frac{2}{3}\frac{k_3'(\alpha)}{\sqrt{T}}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right) - \frac{h(\alpha)\sqrt{1-\alpha^2}}{6\sqrt{T}} - \frac{k_4'(\alpha)}{4!} + \frac{80}{6!}(k_3'(\alpha))^2.$$

Up to $O(T^{-\frac{1}{2}})$ the approximation is

$$I(w) + i(w)\left[\left(\frac{2\alpha}{\sqrt{1-\alpha^2}}\right)\frac{w^2}{\sqrt{T}} - \frac{k_3'(\alpha)}{6}(w^2-1)\right]. \tag{10}$$

In the Appendix we obtain the following explicit representation of $k_3(\alpha)$ up to $O(1)$:

$$k_3(\alpha) = \frac{6T\sigma^6\alpha}{(1-\alpha^2)^2} + O(1),$$

so that, using (4), we have

$$k_3'(\alpha) = \frac{1}{\sqrt{T}}\left(\frac{6\alpha}{\sqrt{1-\alpha^2}}\right) + O(T^{-3/2})$$

and thus (10) becomes

$$I(w) + \frac{i(w)}{\sqrt{T}}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right)(w^2+1). \tag{11}$$

The corresponding approximate density is

$$i(w)\left\{1 + \frac{1}{\sqrt{T}}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right)(w - w^3)\right\}. \tag{12}$$

We notice from both these expressions that the correction term of $O(T^{-\frac{1}{2}})$ increases in magnitude with α. This suggests that the normal approximation is less satisfactory in the less stable case.

3. DISTRIBUTION OF THE t RATIO

In addition to the distribution of $\hat{\alpha}$, we wish also to approximate the sampling distribution of the t ratio test statistic $t = (\hat{\alpha} - \alpha)/s_{\hat{\alpha}}$ where $s_{\hat{\alpha}}^2 = \hat{\sigma}^2/y'C_2y$ and

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(y_t - \hat{\alpha}y_{t-1})^2$$
$$= \frac{1}{T-1}\left\{y'C_3y - \frac{(y'C_1y)^2}{y'C_2y}\right\}$$

where

$$C_3 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Then

$$t = \frac{y'C_1y - \alpha y'C_2y}{\sqrt{\frac{1}{T-1}\left\{(y'C_3y)(y'C_2y) - (y'C_1y)^2\right\}}}$$
$$= \frac{\sqrt{T}(q_1 - \alpha q_2)}{\sqrt{\frac{T}{T-1}\left\{q_3q_2 - q_1^2 + \frac{\mu_3}{T}q_2 + \frac{\mu_2}{T}q_3 - 2\frac{\mu_1}{T}q_1 + \frac{\mu_2}{T}\frac{\mu_3}{T} - \frac{\mu_1^2}{T^2}\right\}}}$$
$$= \sqrt{T}e_T(q), \quad \text{say,}$$

where $q_i = (y'C_iy - \mu_i)/T$, $\mu_i = \operatorname{tr}(C_i\Sigma)$, and $q' = (q_1, q_2, q_3)$.

The function $t = \sqrt{T}e_T(q)$ satisfies the conditions of the theorem in **[6]** and thus the distribution of t admits an Edgeworth series expansion. Since the characteristic function of the q_i is readily available, a convenient algorithm for deriving the expansion is based, as in Section 2, on an expansion of the characteristic function. Here we follow the general procedure outlined in **[8]**.

Writing the derivatives of $e_T(q)$ evaluated at the origin as, for instance $e_j = \partial e_T(0)/\partial q_j$ and using the tensor summation convention, we have from the Taylor development of $e_T(q)$,

$$t = \sqrt{T}(e_jq_j + \tfrac{1}{2}e_{jk}q_jq_k + \tfrac{1}{6}e_{jkl}q_jq_kq_l) + O(T^{-3/2})$$
$$= e_j\bar{q}_j + \frac{1}{2\sqrt{T}}e_{jk}\bar{q}_j\bar{q}_k + \frac{1}{6T}e_{jkl}\bar{q}_j\bar{q}_k\bar{q}_l + O(T^{-3/2}),$$

where $\bar{q}_j = \sqrt{T}q_j$. Now the characteristic function of the $y'C_iy$ is

$$\chi(z) = \left[\det\left\{I - 2i\left(\sum_j z_jC_j\right)\Sigma\right\}\right]^{-\frac{1}{2}}$$

so that the characteristic function of $\bar{q}$ is

$$\theta(z)=\exp\left\{\log\left(\chi\left(\frac{z}{\sqrt{T}}\right)\right)-\frac{i\mu_1 z_1}{\sqrt{T}}-\frac{i\mu_2 z_2}{\sqrt{T}}-\frac{i\mu_3 z_3}{\sqrt{T}}\right\}.$$

The second characteristic of $\bar{q}$ is therefore

$$\begin{aligned}\lambda(z)&=\log(\theta(z))\\&=-\tfrac{1}{2}\log\det\left\{I-\frac{2i}{\sqrt{T}}\left(\sum_j z_j C_j\right)\Sigma\right\}\\&\qquad-\frac{i\mu_1 z_1}{\sqrt{T}}-\frac{i\mu_2 z_2}{\sqrt{T}}-\frac{i\mu_3 z_3}{\sqrt{T}}\end{aligned}$$

and successive derivatives of $\lambda(z)$ evaluated at the origin are given by

$$\lambda_a=\frac{\partial\lambda(0)}{\partial z_a}=0,$$

$$\lambda_{ab}=\frac{\partial^2\lambda(0)}{\partial z_a\partial z_b}=-\frac{2}{T}\operatorname{tr}\{(C_a\Sigma)(C_b\Sigma)\},$$

$$\begin{aligned}\lambda_{abc}=\frac{\partial^3\lambda(0)}{\partial z_a\partial z_b\partial z_c}&=\frac{-4i}{T^{3/2}}\operatorname{tr}\{(C_a\Sigma)(C_b\Sigma)(C_c\Sigma)\}\\&\quad\frac{-4i}{T^{3/2}}\operatorname{tr}\{(C_a\Sigma)(C_c\Sigma)(C_b\Sigma)\},\end{aligned}$$

and

$$\begin{aligned}\lambda_{abcd}=\frac{\partial^4\lambda(0)}{\partial z_a\partial z_b\partial z_c\partial z_d}&=\frac{8}{T^2}[\operatorname{tr}\{(C_d\Sigma)(C_c\Sigma)(C_a\Sigma)(C_b\Sigma)\}\\&\quad+\operatorname{tr}\{(C_d\Sigma)(C_a\Sigma)(C_c\Sigma)(C_b\Sigma)\}+\operatorname{tr}\{(C_d\Sigma)(C_b\Sigma)(C_c\Sigma)(C_a\Sigma)\}\\&\quad+\operatorname{tr}\{(C_b\Sigma)(C_a\Sigma)(C_d\Sigma)(C_c\Sigma)\}+\operatorname{tr}\{(C_b\Sigma)(C_a\Sigma)(C_c\Sigma)(C_d\Sigma)\}\\&\quad+\operatorname{tr}\{(C_a\Sigma)(C_b\Sigma)(C_c\Sigma)(C_d\Sigma)\}].\end{aligned}$$

We now define

$$\begin{aligned}&\alpha_1=\lambda_{jkl}e_je_ke_l, &&\alpha_2=\lambda_{jklm}e_je_ke_le_m,\\&\alpha_3=\gamma_a e_{ab}\gamma_b, &&\alpha_4=\lambda_{ab}e_{ab},\\&\alpha_5=\delta_{ab}e_{ab}, &&\alpha_6=e_{abc}\gamma_a\gamma_b\gamma_c,\\&\alpha_7=e_{abc}\lambda_{ab}\gamma_c, &&\alpha_8=\gamma_b e_{ab}\lambda_{bc}e_{cd}\gamma_d,\\&\alpha_9=\lambda_{ad}e_{ab}\lambda_{bc}e_{cd}, &&\alpha_{10}=\gamma_a e_{ab}\beta_b,\end{aligned}$$

and

$$\begin{aligned}&\omega^2=-\lambda_{jk}e_je_k, &&\gamma_a=\lambda_{ak}e_k,\\&\beta_a=\lambda_{ajk}e_je_k, &&\delta_{ab}=\lambda_{abk}e_k.\end{aligned}$$

Proceeding along the general lines described in [**8**], we then obtain the approximation to $P(t \leqslant x)$ up to $O(T^{-1})$ given by

$$
\begin{aligned}
(13)\qquad & I\left(\frac{x}{\omega}\right)+\left(\frac{\alpha_4}{2\sqrt{T}}\right)\left(\frac{1}{\omega}\right)i\left(\frac{x}{\omega}\right) \\
&+\left(\frac{i\alpha_5}{2\sqrt{T}}+\frac{\alpha_7}{2T}+\frac{\alpha_4^2}{8T}+\frac{\alpha_9}{4T}\right)\left(\frac{1}{\omega^2}\right)i'\left(\frac{x}{\omega}\right) \\
&-\left(\frac{i\alpha_1}{6}+\frac{\alpha_3}{2\sqrt{T}}\right)\left(\frac{1}{\omega^3}\right)i^{(2)}\left(\frac{x}{\omega}\right) \\
&+\left(\frac{\alpha_2}{24}-\frac{i\alpha_{10}}{2\sqrt{T}}-\frac{i\alpha_1\alpha_4}{12\sqrt{T}}-\frac{\alpha_6}{6T}-\frac{\alpha_3\alpha_4}{4T}-\frac{\alpha_8}{2T}\right) \\
&\cdot\left(\frac{1}{\omega^4}\right)i^{(3)}\left(\frac{x}{\omega}\right) \\
&-\left(\frac{\alpha_1^2}{72}-\frac{-i\alpha_1\alpha_3}{12\sqrt{T}}-\frac{\alpha_3^2}{8T}\right)\left(\frac{1}{\omega^6}\right)i^{(5)}\left(\frac{x}{\omega}\right)=I\left(\frac{x}{\omega}\right)+i\left(\frac{x}{\omega}\right) \\
&\cdot\left\{c_o+c_1\left(\frac{x}{\omega}\right)+c_2\left(\frac{x}{\omega}\right)^2+c_3\left(\frac{x}{\omega}\right)^3+c_5\left(\frac{x}{\omega}\right)^5\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
c_0&=\frac{\alpha_4}{2\omega\sqrt{T}}+\frac{i\alpha_1}{6\omega^3}+\frac{\alpha_3}{2\omega^3\sqrt{T}}, \\
c_1&=-\frac{1}{\omega^2}\left(\frac{i\alpha_5}{2\sqrt{T}}+\frac{\alpha_7}{2T}+\frac{\alpha_4^2}{8T}+\frac{\alpha_9}{4T}\right) \\
&\quad+\frac{3}{\omega^4}\left(\frac{\alpha_2}{24}-\frac{i\alpha_{10}}{2\sqrt{T}}-\frac{i\alpha_1\alpha_4}{12\sqrt{T}}-\frac{\alpha_6}{6T}-\frac{\alpha_3\alpha_4}{4T}-\frac{\alpha_8}{2T}\right) \\
&\quad+\frac{15}{\omega^6}\left(\frac{\alpha_1^2}{72}-\frac{i\alpha_1\alpha_3}{12\sqrt{T}}-\frac{\alpha_3^2}{8T}\right), \\
c_2&=-\frac{1}{\omega^3}\left(\frac{i\alpha_1}{6}+\frac{\alpha_3}{2\sqrt{T}}\right), \\
c_3&=-\frac{1}{\omega^4}\left(\frac{\alpha_2}{24}-\frac{i\alpha_{10}}{2\sqrt{T}}-\frac{i\alpha_1\alpha_4}{12\sqrt{T}}-\frac{\alpha_6}{6T}-\frac{\alpha_3\alpha_4}{4T}-\frac{\alpha_8}{2T}\right) \\
&\quad-\frac{10}{\omega^6}\left(\frac{\alpha_1^2}{72}-\frac{i\alpha_1\alpha_3}{12\sqrt{T}}-\frac{\alpha_3^2}{8T}\right),
\end{aligned}
$$

and

$$
c_5=\frac{1}{\omega^6}\left(\frac{\alpha_1^2}{72}-\frac{i\alpha_1\alpha_3}{12\sqrt{T}}-\frac{\alpha_3^2}{8T}\right).
$$

Note that in the above $i\alpha_1$, $i\alpha_5$, and $i\alpha_{10}$ are real in view of the definition of α_1, α_5, and α_{10}, and the form of λ_{jkl}.

An alternative representation of (13) up to $O(T^{-1})$ is

$$I\left(\frac{x}{\omega}+d_o+d_1\left(\frac{x}{\omega}\right)+d_2\left(\frac{x}{\omega}\right)^2+d_3\left(\frac{x}{\omega}\right)^3\right)$$

where $d_0=c_0$, $d_1=c_1+c_0^2/2$, $d_2=c_2$, and $d_3=c_3+c_2c_0$. The expansion up to $O(T^{-\frac{1}{2}})$ is simply

$$I\left(\frac{x}{\omega}\right)+i\left(\frac{x}{\omega}\right)\left\{c_o+c_2\left(\frac{x}{\omega}\right)^2\right\}. \tag{14}$$

From the calculations in the Appendix we obtain the explicit formulae

$$c_o=\frac{1}{\sqrt{T}}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right)\left(\frac{1+3\alpha^2}{1-\alpha^2}\right),$$

$$c_2=\frac{1}{\sqrt{T}}\left(\frac{2\alpha}{\sqrt{1-\alpha^2}}\right)\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^2$$

and

$$\omega^2=1-\frac{1}{T},$$

so that (14) becomes

$$I(x)+\frac{i(x)}{\sqrt{T}}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right)\left\{\frac{1+3\alpha^2}{1-\alpha^2}+2\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^2x^2\right\}+O(T^{-1}). \tag{15}$$

Once again we notice that the correction term of $O(T^{-\frac{1}{2}})$ on the normal approximation increases in magnitude with α. Moreover, comparing (11) and (15) we note the additional factor of $(1/(1-\alpha^2))^2$ in the correction term of (15). This suggests that as α approaches unity, the normal approximation will be even less satisfactory for the t ratio than for $\hat{\alpha}$.[2]

4. NAGAR-TYPE APPROXIMATIONS

An alternative approximation which has been found useful for comparative purposes [**9**] is the normal distribution with first and second moments given by the Nagar approximations. These approximations to the moments can be derived from the Edgeworth expansion above as in [**9**].

[2] In the original version of this paper [**7**] the explicit representation of the expansion given by (15) had not been derived. In the numerical computations I then reported (in [**7**]) the t ratio expansion was at that stage calculated using (14) and the general formulae for the c_i and α_i given earlier in this section. I have since discovered that a programming error in a loop of the program led to incorrect results in the second run through the loop for the larger parameter value $\alpha=0.8$, although computations for $\alpha=0.4$ in the first run through the loop were correct. These incorrect results led to the reverse conclusion in [**7**] that as α approached unity the normal approximation became more, rather than less, satisfactory.

We introduce ξ_T to represent $\sqrt{T}(\hat{\alpha}-\alpha)/\sqrt{1-\alpha^2}$ in Section 2 and t/ω in Section 3. We have, in general,

$$P(\xi_T \leq w) = I(w + g_o + g_1 w + g_2 w^2 + g_3 w^3) + O(T^{-3/2})$$

and introducing $d = g_0 + (1+g_1)w + g_2 w^2 + g_3 w^3$ we find that the critical level of ξ_T corresponding to a given probability $p = I(d)$ is given by

$$\xi_T = -g_0 + (1 - g_1 + 2g_0 g_2)d - g_2 d^2 + (2g_2^2 - g_3)d^3 + R \tag{16}$$

where $R = O(T^{-3/2})$. We can, in fact, regard (16) as a transformation from a variate d with a standard normal distribution to the variate ξ_T (cf. [**9**]). Taking expectations and assuming $E(R) = O(T^{-3/2})$ we have

$$E(\xi_T) = -g_0 - g_2 + O(T^{-3/2}). \tag{17}$$

Similarly,

$$\begin{aligned} E(\xi_T^2) &= g_0^2 + (1 - g_1 + 2g_0 g_2)^2 E(d^2) + g_2^2 E(d^4) \\ &\quad + 2g_0 g_2 E(d^2) + 2(2g_2^2 - g_3)E(d^4) + O(T^{-3/2}) \\ &= 1 + g_0^2 - 2g_1 + 6g_0 g_2 + 15g_2^2 - 6g_3 + O(T^{-3/2}). \end{aligned} \tag{18}$$

The validity of these expansions will not be examined in detail here but we note that for the case $\xi_T = \sqrt{T}(\hat{\alpha}-\alpha)/\sqrt{1-\alpha^2}$ (17) becomes

$$\begin{aligned} E\left(\frac{\sqrt{T}(\hat{\alpha}-\alpha)}{\sqrt{1-\alpha^2}}\right) &= -\frac{k_3'(\alpha)}{6} - \left(\frac{2\alpha}{\sqrt{1-\alpha^2}}\right)\frac{1}{\sqrt{T}} + \frac{k_3'(\alpha)}{6} + O(T^{-3/2}) \\ &= -\left(\frac{2\alpha}{\sqrt{1-\alpha^2}}\right)\frac{1}{\sqrt{T}} + O(T^{-3/2}) \end{aligned}$$

which accords with earlier results [**4** and **10**].

5. NUMERICAL COMPARISONS

Since the exact distribution of $\hat{\alpha}$ can be calculated by numerical integration[3] it is possible to assess the accuracy of the approximations to $O(T^{-\frac{1}{2}})$ and $O(T^{-1})$ developed in Section 2. Various values of T and α were selected and in Figures 1–4 we graph the exact distribution function of $\sqrt{T}(\hat{\alpha}-\alpha)/\sqrt{1-\alpha^2}$ against the corresponding approximate distribution to $O(T^{-\frac{1}{2}})$ given by (11) and the asymptotic normal approximation.

We notice from Figures 1–4 that the exact distribution is negatively skewed and downward biased. These features are more accentuated for $\alpha = 0.8$ than for $\alpha = 0.4$ and the normal approximation is, therefore, less satisfactory in the less stable case.[4] For $\alpha = 0.4$ the $O(T^{-\frac{1}{2}})$ approximation captures the location of the

[3] The procedure employed here was Imhof's numerical inversion of the characteristic function of a quadratic form in normal variates [**5**].

[4] This confirms what was earlier implied by the correction term of $O(T^{-\frac{1}{2}})$ in (11) at the end of Section 2.

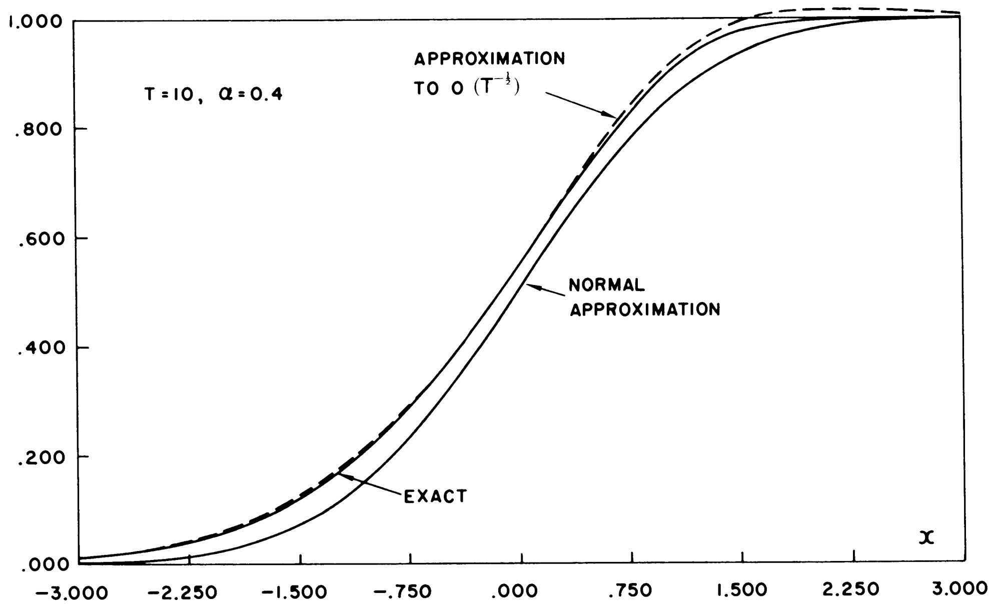

FIGURE 1—$T = 10$, $\alpha = 0.4$.

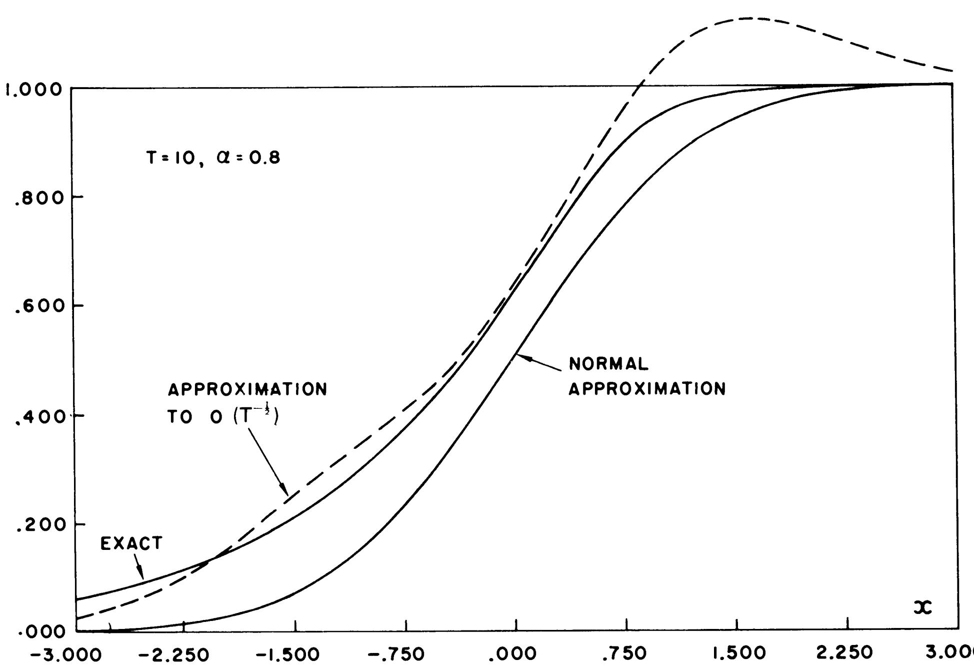

FIGURE 2—$T = 10$, $\alpha = 0.8$.

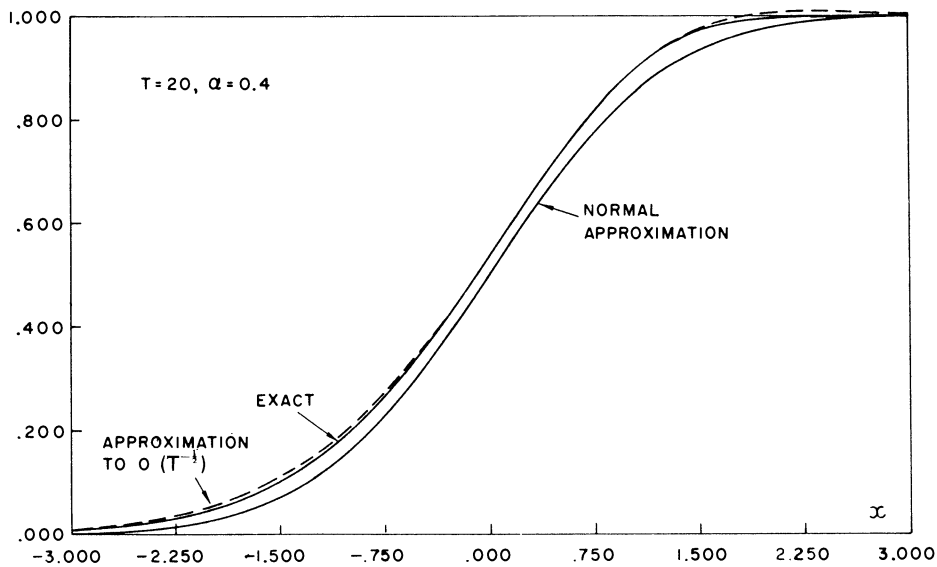

FIGURE 3—$T = 20, \alpha = 0.4$.

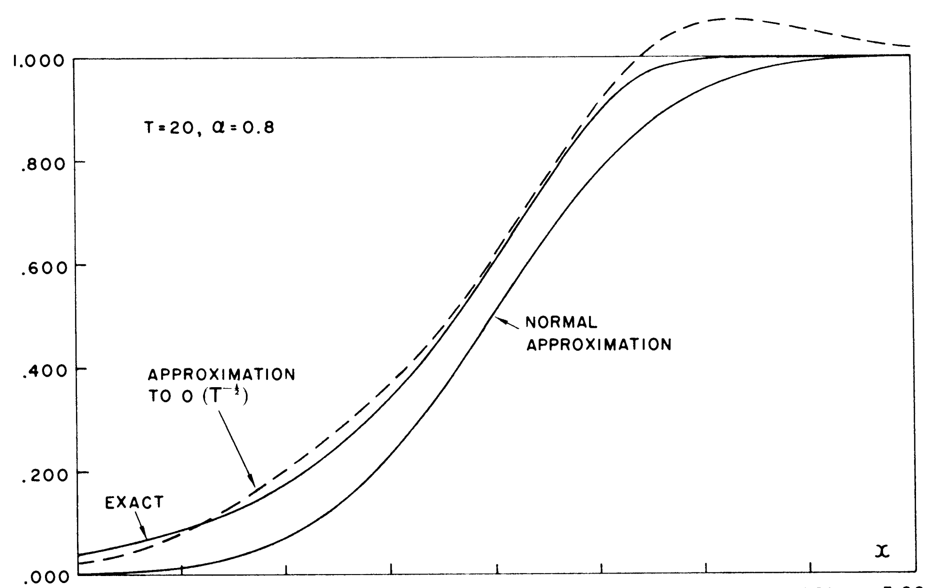

FIGURE 4—$T = 20, \alpha = 0.8$.

distribution well and also gives a good approximation to the long left-hand tail even for $T = 10$. In places the $O(T^{-\frac{1}{2}})$ approximation overcorrects in the body of the distribution around $x = -1.5$ and in the right-hand tail where it overshoots unity. For $T = 20$ and $\alpha = 0.4$, the $O(T^{-\frac{1}{2}})$ approximation is very close indeed to the exact distribution.

When $\alpha = 0.8$ the $O(T^{-\frac{1}{2}})$ approximation is much less satisfactory. For the smaller sample size ($T = 10$) the error in the approximation is, in places, considerable and the overcorrection on the normal approximation error results in the $O(T^{-\frac{1}{2}})$ approximation having the greater error in certain parts of the distribution. Moreover, when T increases the error in the $O(T^{-\frac{1}{2}})$ approximation does not disappear as quickly as in the case of $\alpha = 0.4$. For $T = 20$ the $O(T^{-\frac{1}{2}})$ approximation is generally much closer to the exact distribution than the normal but it fails to pick up the left-hand tail as well as in the more stable case and still overshoots unity rather early.

The $O(T^{-1})$ approximation to the distribution of $\sqrt{T}(\hat{\alpha} - \alpha)/\sqrt{1 - \alpha^2}$ has also been calculated for similar values of T and α and in Figures 5 and 6 we compare

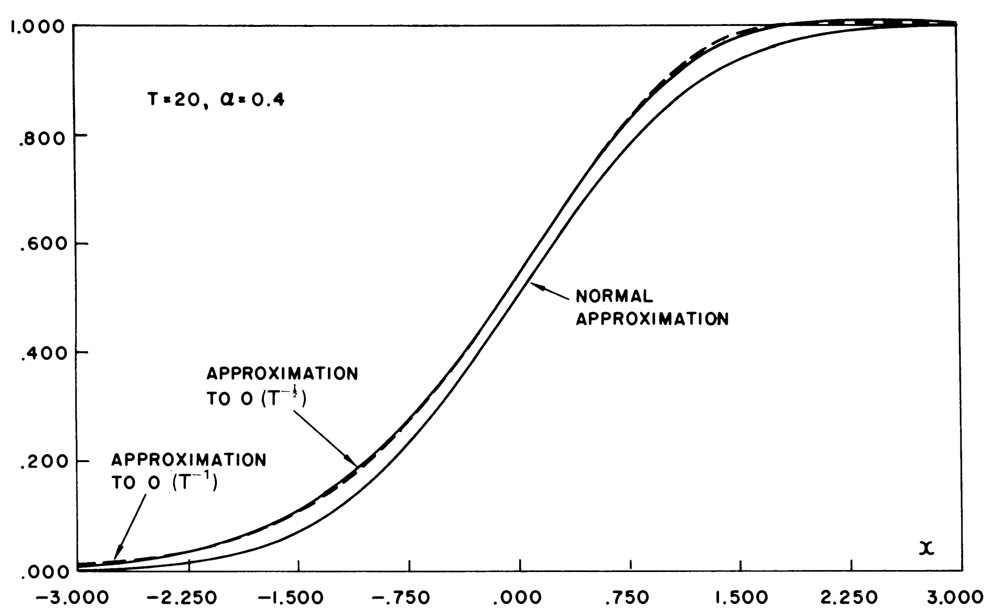

FIGURE 5—$T = 20, \alpha = 0.4$.

this higher order approximation with the $O(T^{-\frac{1}{2}})$ and normal approximations when $T = 20$.[5] For $\alpha = 0.4$ we see from the figures that the $O(T^{-1})$ and $O(T^{-\frac{1}{2}})$ approximations are close together; and where they do differ the $O(T^{-1})$ approximation appears to be correcting in those parts of the distribution where the $O(T^{-\frac{1}{2}})$ approximation initially overcorrected on the normal approximation; that is, in the region around $x = -1.5$ and in the upper right-hand tail.

[5] Comparisons for other values of T are recorded in [**7**].

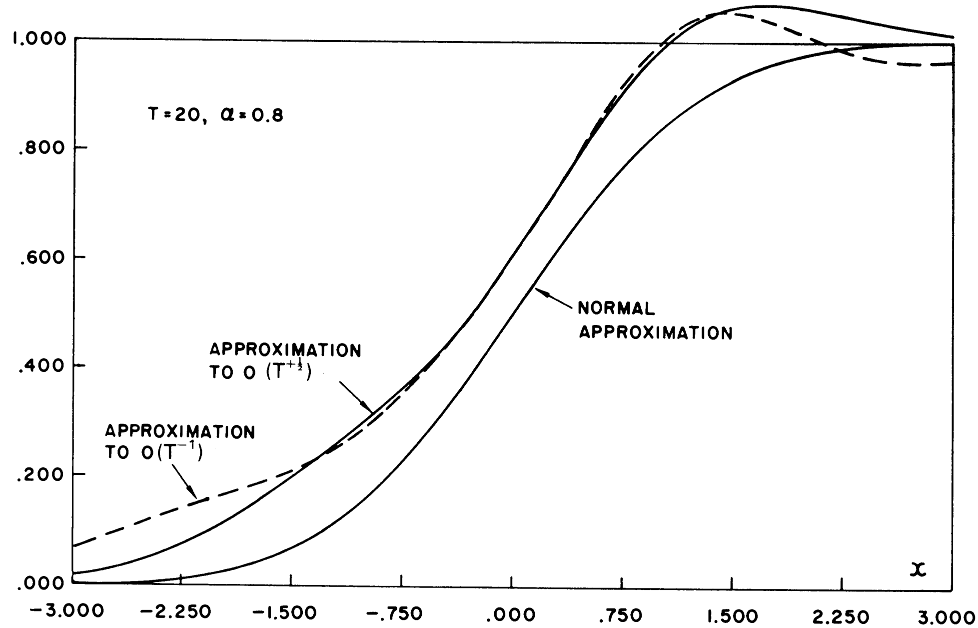

FIGURE 6—$T = 20, \alpha = 0.8$.

In the less stable case when $\alpha = 0.8$ the $O(T^{-1})$ approximation overcorrects badly on the $O(T^{-\frac{1}{2}})$ approximation. This overcorrection occurs mainly in the tails and, although it is moderated as T increases, it is still sufficiently large for $T = 20$ to seriously distort the tail area probabilities.

Tail area probabilities have been calculated for the distribution of $\hat{\alpha}$ where direct comparisons can be made between the approximate and exact probabilities. In Tables I–IV we record the exact and approximate values of $P(|\sqrt{T}(\hat{\alpha} -$

TABLE I

TAIL PROBABILITIES: $T = 10$, $\alpha = 0.4$

X	Exact	Approximation to $O(T^{-1})$	Nagar Approximation	Normal
1.95	0.0546	0.0531	0.0619	0.0511
2.00	0.0498	0.0496	0.0555	0.0455
2.05	0.0453	0.0466	0.0496	0.0403
2.10	0.0412	0.0439	0.0433	0.0357
2.15	0.0375	0.0416	0.0395	0.0315
2.20	0.0340	0.0359	0.0351	0.0278
2.25	0.0309	0.0377	0.0311	0.0244
2.30	0.0279	0.0361	0.0275	0.0214
2.35	0.0253	0.0346	0.0243	0.0187
2.40	0.0228	0.0332	0.0214	0.0163
2.45	0.0206	0.0318	0.0189	0.0142
2.50	0.0185	0.0306	0.0165	0.0124
2.55	0.0166	0.0293	0.0145	0.0107
2.60	0.0149	0.0281	0.0127	0.0093

TABLE II

TAIL PROBABILITIES: $T=10$, $\alpha=0.8$

X	Exact	Approximation to $O(T^{-1})$	Nagar Approximation	Normal
1.95	0.1363	0.2661	0.3015	0.0511
2.00	0.1302	0.2725	0.2891	0.0455
2.05	0.1244	0.2175	0.2771	0.0403
2.10	0.1188	0.2699	0.2654	0.0357
2.15	0.1134	0.2677	0.2540	0.0315
2.20	0.1083	0.2649	0.2430	0.0278
2.25	0.1034	0.2613	0.2323	0.0244
2.30	0.0987	0.2570	0.2220	0.0214
2.35	0.0942	0.2521	0.2119	0.0187
2.40	0.0899	0.2464	0.2022	0.0163
2.45	0.0858	0.2400	0.1928	0.0142
2.50	0.0819	0.2330	0.1837	0.0124
2.55	0.0781	0.2255	0.1750	0.0107
2.60	0.0745	0.2174	0.1665	0.0093

$\alpha)/\sqrt{1-\alpha^2}|>X)$ in the region $1.95\leq X\leq 2.60$ for $T=10$, $T=30$, and $\alpha=0.4$, 0.8. In addition to the $O(T^{-1})$ approximation derived from (9), we give the tail probabilities of the Nagar approximation described in Section 4 and those of the asymptotic normal approximation. One feature of the results which stands out is the extent to which the $O(T^{-1})$ approximation overestimates the symmetric tail probabilities for the less stable case $\alpha=0.8$. The Nagar approximation is even worse for moderate X values, but improves on the $O(T^{-1})$ results for large X. On the other hand, for $\alpha=0.4$ the $O(T^{-1})$ approximation gives much better results and is, for $T=30$, generally closer to the exact values than is the Nagar approximation.

TABLE III

TAIL PROBABILITIES: $T=30$, $\alpha=0.4$

X	Exact	Approximation to $O(T^{-1})$	Nagar Approximation	Normal
1.95	0.0511	0.0503	0.0503	0.0511
2.00	0.0460	0.0453	0.0472	0.0455
2.05	0.0414	0.0409	0.0419	0.0403
2.10	0.0373	0.0369	0.0372	0.0357
2.15	0.0335	0.0333	0.0329	0.0315
2.20	0.0302	0.0301	0.0290	0.0278
2.25	0.0272	0.0272	0.0255	0.0244
2.30	0.0245	0.0247	0.0224	0.0214
2.35	0.0220	0.0224	0.0197	0.0187
2.40	0.0198	0.0204	0.0172	0.0163
2.45	0.0178	0.0185	0.0150	0.0142
2.50	0.0161	0.0169	0.0131	0.0124
2.55	0.0144	0.0154	0.0113	0.0107
2.60	0.0130	0.0141	0.0098	0.0093

TABLE IV

TAIL PROBABILITIES: $T=30$, $\alpha=0.8$

X	Exact	Approximation to $O(T^{-1})$	Nagar Approximation	Normal
1.95	0.1007	0.1157	0.1341	0.0511
2.00	0.0955	0.1146	0.1243	0.0455
2.05	0.0905	0.1137	0.1151	0.0403
2.10	0.0857	0.1128	0.1064	0.0357
2.15	0.0812	0.1118	0.0982	0.0315
2.20	0.0769	0.1089	0.0906	0.0278
2.25	0.0728	0.1045	0.0834	0.0244
2.30	0.0689	0.1002	0.0767	0.0214
2.35	0.0652	0.0958	0.0704	0.0187
2.40	0.0617	0.0915	0.0646	0.0163
2.45	0.0584	0.0872	0.0591	0.0142
2.50	0.0552	0.0830	0.0541	0.0124
2.55	0.0522	0.0787	0.0494	0.0107
2.60	0.0493	0.0745	0.0450	0.0093

In some cases the Nagar approximation does well and is frequently a good deal better than the normal approximation. However, the Nagar approximation performs poorly relative to the $O(T^{-1})$ approximation in the region of $X=2.00$ for all values of T and α.

Clearly, the overestimation of the tail probabilities by the $O(T^{-1})$ approximation is the result of the overcorrection of this approximation in the tails of the distribution. Some efforts were made to overcome this problem and improve on the results of the $O(T^{-1})$ approximation given in the tables. The alternative representation of the distribution function as

$$I(X+b_0+b_1X+b_2X^2+b_3X^3) \tag{19}$$

(given in Section 2) was tried since it has the useful property that it lies in the [0, 1] interval. But this form of the approximation gave poor results with very thin tails in the case of $\alpha=0.8$.[6] Monotonic approximations derived from (19) along the lines of Sargan and Mikhail [**9**] also gave poor results.

6. CONCLUSION

One of the main points to stand out from the computations reported in the previous section is the fact that the stability of the model is of importance in the performance of the Edgeworth expansion. In general, for a less stable model we seem to get a less satisfactory representation of the finite sample distribution of the least squares estimator. Intuitively, this is likely to be tied up with the fact that when $|\alpha|\geq 1$ the limiting distribution of $\hat{\alpha}$ is no longer normal. When $|\alpha|>1$ the

[6] This appears to be the result of the signs of the coefficients b_3 in (19) and a_5 in (9). In particular, in the cases considered, a_5 was negative and b_3 positive. With $a_5<0$ the left-hand tail of (9) is fattened for large (negative) X, while with $b_3>0$ the left-hand tail of (19) is thinned out for large (negative) X.

limiting distribution of $\hat{\alpha}$, suitably normalized, is Cauchy while for $|\alpha|=1$ the distribution is not known in closed form although it is known not to be normal **[11]**.

In the case of the t ratio we have not computed the exact distribution so we cannot directly confirm that the performance of the Edgeworth expansion is also less satisfactory in the less stable case. However, it seems likely that this will be so. For, the explicit representation of this expansion to $O(T^{-\frac{1}{2}})$ given by (15) indicates that convergence to the normal approximation with increasing T is even slower as α approaches unity for the t ratio than for $\hat{\alpha}$.[7] Moreover, although the t ratio has a limiting normal distribution for all values of α satisfying $|\alpha|>1$ as well as $|\alpha|<1$, the asymptotic distribution has not been determined in the case $|\alpha|=1$ and this case is likely to be the one exception to the usual theory **[12]**.

In the Introduction we suggested that our results in this simple autoregressive model should help us to assess the usefulness of the Edgeworth approximation in more general dynamic models. It turns out, however, that our results are not as unambiguous as we would have liked. On the one hand, our computations in Section 5 tend to confirm the general conclusion in **[2]** that asymptotic theory does not provide a good guide to the finite sample distribution of estimators and test statistics in dynamic models. In addition, we find encouragement in that good approximations can be obtained by the Edgeworth expansion in some cases, even for quite small sample sizes. But, on the other hand, our results suggest caution in that the reliability of the approximations does seem sensitive to the stability of the model and, when the approximations are unreliable, tail area probabilities can be badly distorted. Further work seems desirable on a number of these points.

University of Essex

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APPENDIX

The purpose of this appendix is to indicate how explicit representations of the expansions given in Sections 2 and 3 of the paper can be derived and to detail these derivations in the case of the expansions up to $O(T^{-\frac{1}{2}})$. Taking the distribution of $\hat{\alpha}$ first we see that the coefficients a_i $(i=1,\ldots,5)$ in (9) depend on $k_s(r)$ and its derivatives, both evaluated at $r=\alpha$. In particular we need $k_3(\alpha)$, $\partial k_3(\alpha)/\partial r$, and $k_4(\alpha)$. Then

$$k_3'(\alpha)=\frac{k_3(\alpha)}{(k_2(\alpha))^{3/2}},$$

$$\frac{\partial k_3'(\alpha)}{\partial r}=\frac{\partial k_3(\alpha)/\partial r}{(k_2(\alpha))^{3/2}}-\frac{3}{2}\frac{k_3(\alpha)\partial k_2(\alpha)/\partial r}{(k_2(\alpha))^{5/2}},$$

and

$$k_4'(\alpha)=\frac{k_4(\alpha)}{(k_2(\alpha))^{2}}.$$

[7] As I indicated in Footnote 2, an error in the program for the computation of the t ratio expansion from the general formula (14) led to the reverse conclusion in the original version of this paper **[7]**.

From the general formula in Section 2 we have

$$k_s(r)=(s-1)!2^{s-1}\left(\frac{T}{2\pi}\right)\sum_{j=0}^{s}\binom{s}{j}(-r)^{s-j}\int_{-\pi}^{\pi}(2\pi f(\lambda))^s(\cos\lambda)^j\,d\lambda+O(1). \tag{20}$$

Thus, $k_s(r)$ is a polynomial in r whose coefficients can be determined up to $O(1)$ by evaluating the integrals

$$\int_{-\pi}^{\pi}(2\pi f(\lambda))^s\cos(j\lambda)\,d\lambda \tag{21}$$

for various integers j; and, using standard trigonometric formulae, we can then obtain

$$\int_{-\pi}^{\pi}(2\pi f(\lambda))^s(\cos\lambda)^j\,d\lambda.$$

For even quite small s ($s=3,4$) the algebra involved is exceedingly heavy. We note, however, that since

$$(2\pi f(\lambda))^s=\frac{\sigma^{2s}}{|(1-\alpha\, e^{i\lambda})^s|^2},$$

(21) can be evaluated using residue theory as follows:

$$\begin{aligned}\int_{-\pi}^{\pi}(2\pi f(\lambda))^s\cos(j\lambda)\,d\lambda&=\sigma^{2s}\int_{-\pi}^{\pi}\frac{e^{ij\lambda}\,d\lambda}{|(1-\alpha\, e^{i\lambda})^s|^2}\\&=\frac{\sigma^{2s}}{i}\int_{|z|=1}\frac{z^{j+s-1}\,dz}{(1-\alpha z)^s(z-\alpha)^s}\\&=\sigma^{2s}2\pi\left\{\text{Residue of }\frac{z^{j+s-1}}{(1-\alpha z)^s(z-\alpha)^s}\text{ at }z=\alpha\right\}\\&=\sigma^{2s}2\pi\lim_{z\to\alpha}\left\{\frac{1}{(s-1)!}\frac{d^{s-1}}{dz^{s-1}}\left(\frac{z^{j+s-1}}{(1-\alpha z)^s}\right)\right\}.\end{aligned}$$

When $s=3$ we obtain

$$\begin{aligned}\int_{-\pi}^{\pi}(2\pi f(\lambda))^3\cos(j\lambda)\,d\lambda&=(\sigma^6 2\pi)\left(\frac{\alpha^j}{2(1-\alpha^2)^3}\right)\\&\quad\cdot\left[\frac{2+8\alpha^2+2\alpha^4}{(1-\alpha^2)^2}+\frac{3(1+\alpha^2)j}{1+\alpha^2}+j^2\right]\end{aligned}$$

and then, after more algebra, we find

$$\int_{-\pi}^{\pi}(2\pi f(\lambda))^3\,d\lambda=2\pi\sigma^6\left(\frac{2+8\alpha^2+2\alpha^4}{2(1-\alpha^2)^5}\right),$$

$$\int_{-\pi}^{\pi}(2\pi f(\lambda))^3\cos\lambda\,d\lambda=2\pi\sigma^6\left(\frac{3\alpha(1+\alpha^2)}{(1-\alpha^2)^5}\right),$$

$$\int_{-\pi}^{\pi}(2\pi f(\lambda))^3\cos^2\lambda\,d\lambda=2\pi\sigma^6\left(\frac{1+10\alpha^2+\alpha^4}{2(1-\alpha^2)^5}\right),$$

and

$$\int_{-\pi}^{\pi}(2\pi f(\lambda))^3\cos^3\lambda\,d\lambda=2\pi\sigma^6\left(\frac{\alpha(9+19\alpha^2-5\alpha^4+\alpha^6)}{4(1-\alpha^2)^5}\right).$$

Returning to (20) we find for $s=3$

$$\begin{aligned}k_3(r)&=\frac{8T\sigma^6}{(1-\alpha^2)^5}\\&\quad\cdot\left\{\frac{\alpha}{4}(9+19\alpha^2-5\alpha^4+\alpha^6)-\frac{3}{2}r(1+10\alpha^2+\alpha^4)\right.\\&\quad\left.+9r^2\alpha(1+\alpha^2)-r^3(1+4\alpha^2+\alpha^4)\right\}+O(1)\end{aligned} \tag{22}$$

so that

$$k_3(\alpha)=\left(\frac{8T\sigma^6}{4(1-\alpha^2)^5}\right)3\alpha(1-\alpha^2)^3+O(1)=\frac{6T\sigma^6\alpha}{(1-\alpha^2)^2}+O(1).$$

This enables us to obtain an explicit representation of the $O(T^{-\frac{1}{2}})$ expansion in Section 2. To do the same for the expansion up to $O(T^{-1})$ we need $\partial k_3(\alpha)/\partial r$, which can be readily deduced from (22) above, and $k_4(\alpha)$, which can be obtained in the same way as $k_3(\alpha)$.

Turning now to the t ratio, we note that the expansion up to $O(T^{-\frac{1}{2}})$ depends on derivatives of $e_T(q)$ up to the second order and derivatives of $\lambda(z)$ up to the third order (both evaluated at the origin). First of all, we see that

$$\mu_1=\frac{T\sigma^2\alpha}{1-\alpha^2},\qquad \mu_2=\frac{T\sigma^2}{1-\alpha^2},\qquad \mu_3=\frac{T\sigma^2}{1-\alpha^2},$$

and then

$$\mu_2\mu_3-\mu_1^2=\frac{T^2\sigma^4}{1-\alpha^2}.$$

Hence,

$$e_1=\frac{1}{\sigma^2}\sqrt{\frac{T-1}{T}(1-\alpha^2)},\qquad e_2=-\frac{\alpha}{\sigma^2}\sqrt{\frac{T-1}{T}(1-\alpha)^2},\qquad e_3=0,$$

$$e_{11}=-\frac{2\alpha}{\sigma^4}\sqrt{\frac{T-1}{T}(1-\alpha^2)},\qquad e_{12}=e_{21}=-\frac{1+2\alpha^2}{2\sigma^4}\sqrt{\frac{T-1}{T}(1-\alpha^2)},$$

$$e_{13}=e_{31}=-\frac{1}{2\sigma^4}\sqrt{\frac{T-1}{T}(1-\alpha^2)},\qquad e_{22}=\frac{\alpha}{\sigma^4}\sqrt{\frac{T-1}{T}(1-\alpha^2)},$$

$$e_{23}=e_{32}=\frac{\alpha}{2\sigma^4}\sqrt{\frac{T-1}{T}(1-\alpha^2)},\qquad e_{33}=0.$$

Now $\omega^2=-\lambda_{jk}e_je_k$ and

$$\lambda_{jk}=-\frac{2}{T}\operatorname{tr}\{(C_j\Sigma)(C_k\Sigma)\}$$

so that

$$\begin{aligned}\omega^2&=\frac{2}{T}\left(\frac{1}{\sigma^2}\sqrt{\frac{T-1}{T}(1-\alpha^2)}\right)^2\operatorname{tr}((C_1-\alpha C_2)\Sigma)^2\\&=\frac{2}{T}\left(\frac{1}{\sigma^4}\frac{T-1}{T}(1-\alpha^2)\right)\frac{T}{2}\frac{\sigma^4}{1-\alpha^2}\\&=1-\frac{1}{T}\end{aligned}$$

Setting $\eta_{jkl}=i\lambda_{jkl}$ we also have

$$i\alpha_1=i\lambda_{jkl}e_je_ke_l=\eta_{jkl}e_je_ke_l$$

and

$$\eta_{111}=\frac{8}{T^{3/2}}\operatorname{tr}(C_1\Sigma)^3,$$

$$\eta_{121}=\eta_{211}=\eta_{112}=\frac{8}{T^{3/2}}\operatorname{tr}\{(C_1\Sigma)^2C_2\Sigma\},$$

$$\eta_{122}=\eta_{221}=\eta_{212}=\frac{8}{T^{3/2}}\operatorname{tr}\{C_1\Sigma(C_2\Sigma)^2\},$$

$$\eta_{222}=\frac{8}{T^{3/2}}\operatorname{tr}(C_2\Sigma)^3,$$

so that

$$i\alpha_1 = \frac{8}{T^{3/2}}\frac{1}{\sigma^6}\left(\frac{T-1}{T}(1-\alpha^2)\right)^{3/2} \operatorname{tr}((C_1-\alpha C_2)\Sigma)^3$$
$$= \frac{8}{T^{3/2}}\frac{1}{\sigma^6}\left(\frac{T-1}{T}(1-\alpha^2)\right)^{3/2}\frac{1}{8}\left(\frac{6T\sigma^6\alpha}{(1-\alpha^2)^2}\right) + O(T^{-3/2})$$
$$= \frac{6}{\sqrt{T}}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right) + O(T^{-3/2}).$$

We find after further algebra along the same lines that

$$\alpha_3 = (\lambda_{aj}e_j)e_{ab}(\lambda_{bk}e_k)$$
$$= -\left(\frac{2\alpha}{\sqrt{1-\alpha^2}}\right)\left(\frac{3+2\alpha^2+3\alpha^4}{(1-\alpha^2)^2}\right) + O(T^{-1})$$

and

$$\alpha_4 = \lambda_{jk}e_{jk}$$
$$= \left(\frac{2\alpha}{\sqrt{1-\alpha^2}}\right)\left(\frac{3+6\alpha^2-\alpha^4}{(1-\alpha^2)^2}\right) + O(T^{-1}).$$

Thus, the coefficients in the expansion (14) of the paper are given by

$$c_0 = \frac{i\alpha_1}{6} + \frac{1}{2\sqrt{T}}(\alpha_3+\alpha_4) = \frac{1}{\sqrt{T}}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right)\left(\frac{1+3\alpha^2}{1-\alpha^2}\right) + O(T^{-3/2})$$

and

$$c_2 = -\left(\frac{i\alpha_1}{6} + \frac{\alpha_3}{2\sqrt{T}}\right) = \frac{1}{\sqrt{T}}\left(\frac{2\alpha}{\sqrt{1-\alpha^2}}\right)\left(\frac{1+\alpha^2}{1+\alpha^2}\right)^2 + O(T^{-3/2}).$$

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