CHAPTER 8

*Some Computations Based on Observed Data Series of the Exogenous Variable Component in Continuous Systems**

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1. Introduction

One of the results of my earlier paper [2] indicated that the order of magnitude of the asymptotic bias of estimators derived from the exact discrete model that corresponds to a structural system of stochastic differential equations depends on the smoothness properties of the exogenous variable series. For instance, if the exogenous variables are non-random, uniformly bounded functions of time with continuous derivatives to the third order that are also uniformly bounded then the asymptotic bias has a smaller order of magnitude in terms of the sampling interval than it would if the derivatives of the functions defining the exogenous variables have discontinuities at a countable set of isolated points on the real line. We can expect this result concerning asymptotic bias to have practical implications in empirical work with finite samples. A sampling experiment reported in [3] already indicates that, when the the exogenous series are generated by a first-order system of stochastic differential equations driven by pure noise (so that the exogenous series are the realisations of processes whose second spectral moments do not exist and are therefore no smoother than the endogenous series), then estimates from the exact discrete model are somewhat disappointing in view of the greater computational burden involved in estimating this model rather than the discrete approximation.

Hence, the question of what assumptions are realistic about the

* The paper is based on chapter 6 of my thesis [3]. I am very grateful to Professor J. D. Sargan who first suggested to me that some computations with observed data series might be useful in this context. I wish also to thank Mrs. P. Kurukulaaratchy for her help in my programming work and Mrs. Rachel Britton and Mrs. Jillian Smith for their assistance in preparing the charts in this paper.

exogenous variables does seem to be of some importance. Our purpose in this paper is to tackle this question, but we do so in a rather indirect way. We first take a number of observed data series of typical exogenous variables and select the time unit in such a way that we have a reasonable number of intermediate observations. Thus, we may specify the unit of time to be a quarter when, in fact, we have weekly observations. The extra observations enable us to compute more or less exactly the exogenous variable component in the exact model; we can then consider how good the approximation implicit in the exact model (see [2]) and the discrete approximation (see [1], [5] and [6]) are for this particular series. We carry out these computations for a large number of different eigenvalues (both real and complex) so that we can determine whether or not the performance of the approximations appears to be sensitive to the size of the system eigenvalues.

2. *The exogenous variable component and its approximations*

The exogenous variable component in the exact discrete model corresponding to a linear system of stochastic differential equations is known to be of the form [1]

$$\int_0^h \exp(sA)Bz(th - s)\,\mathrm{d}s, \tag{1}$$

where A and B are the coefficient matrices in the structural system and z is the vector of exogenous variables. We specify the dimensions of A and B to be $n \times n$ and $n \times m$, respectively.

Taking the eigenvalues of A to be distinct, so that there exists a non-singular matrix T which reduces A by a similarity transformation to the diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, we can write the ith element of (1) as

$$\sum_j \sum_k \sum_l t_{ij} t^{jk} b_{kl} \int_0^h \exp(s\lambda_j) z_l(th - s)\,\mathrm{d}s, \tag{2}$$

where $T = (t_{ij})$, $T^{-1} = (t^{ij})$ and $B = (b_{ij})$. The implication of expression (2) is that provided we can obtain a good approximation to

[1] See [2] and [5].

$$\Phi_{tjl} = \int_0^h \exp(s\lambda_j) z_l(th - s)\,\mathrm{d}s, \tag{3}$$

for all j and l, our approximation to (1) will be a good one.

We already know that the discrete approximation implies an approximation to (1) which has the form

$$(I - \tfrac{1}{2}hA)^{-1} hB\{z(th) + z(th - h)\}/2, \tag{4}$$

and that the exact model approximation to (1) given in [2] is[2]

$$E_2 z(th) + E_3 z(th - h) + E_4 z(th - 2h), \tag{5}$$

where the elements of E_2, E_3 and E_4 are non-linear functions[3] of the elements at A and B.

The ith elements of (4) and (5) can be written

$$\sum_j \sum_k \sum_l t_{ij} t^{jk} b_{kl} \Phi^D_{tjl}, \tag{6}$$

and

$$\sum_j \sum_k \sum_l t_{ij} t^{jk} b_{kl} \Phi^E_{tjl}, \tag{7}$$

where

$$\Phi^D_{tjl} = h(1 - \tfrac{1}{2}h\lambda_j)^{-1}\{z_l(th) + z_l(th - h)\}/2, \tag{8}$$

and

$$\begin{aligned} \Phi^E_{tjl} = {} & h(h\lambda_j)^{-3} \\ & \times \{[\{1 + \tfrac{1}{2}(h\lambda_j)\}\exp(h\lambda_j) - (h\lambda_j)^2 - 3(h\lambda_j)/2 - 1]z(th) \\ & + [\{(h\lambda_j)^2 - 2\}\exp(h\lambda_j) + 2(h\lambda_j) + 2]z(th - h) \\ & + [\{1 - \tfrac{1}{2}(h\lambda_j)\}\exp(h\lambda_j) - \tfrac{1}{2}(h\lambda_j) - 1]z(th - 2h)\}. \end{aligned} \tag{9}$$

It follows from (2), (6) and (7) that the discrete approximation and the exact model imply as approximations to the integral Φ_{tjl} given by (3) the expressions Φ^D_{tjl} and Φ^E_{tjl}, respectively.

If we are prepared to make certain assumptions about the exogenous

[2] This approximation is obtained by replacing $z(th - s)$ in (1) by a three-point Lagrange interpolation formula which passes through the three consecutive observations $z(th - 2h)$, $z(th - h)$ and $z(th)$.

[3] These functions are stated in full in [2] and in Chapter 7 of this book.

variables, then we can analyse the specification errors that result from the use of the approximations (8) and (9). This type of analysis has been carried out in Sargan [5] and Phillips [2]. When z has continuous, uniformly bounded derivatives to the third order we know that [4] (dropping the subscripts j and l)

$$\Phi_t - \Phi_t^D = -(1 - \tfrac{1}{2}h\lambda)^{-1}\{\tfrac{1}{12}h^3(\lambda^2 z_t + \lambda z_t^{(1)} + z_t^{(2)}) + O(h^4)\}, \tag{10}$$

and

$$\Phi_t - \Phi_t^E = -\tfrac{1}{6}\int_0^h e^{s\lambda}\{s(h-s)(2h-s)\}z^{(3)}(\theta)\,ds, \tag{11}$$

where θ in the integrand on the right side of (11) is an unknown function of s and satisfies $th - 2h < \theta < th$.

For our purposes in this paper we make two observations on these expressions for the errors in the approximations. First, we see from (10) and (11) that for an exogenous series satisfying the stated conditions the moduli of the errors are

$$|\Phi_t - \Phi_t^D| = O(h^3) \quad \text{and} \quad |\Phi_t - \Phi_t^E| = O(h^4), \tag{12}$$

uniformly in t. For small h, (12) suggests that (9) will be a better approximation than (8).

Our second observation is that for fixed h the approximation Φ_t^E will be more reliable than Φ_t^D when we allow the system eigenvalue to take any value in the left half plane. To show this, we first define

$$\begin{aligned}\varepsilon(t, h, \lambda) &= (1 - \tfrac{1}{2}h\lambda)(\Phi_t - \Phi_t^D) \\ &= (1 - \tfrac{1}{2}h\lambda)\int_0^h e^{s\lambda}z(th - s)\,ds - \tfrac{1}{2}h\{z(th) + z(th - h)\}.\end{aligned}$$

As the real part of λ tends to minus infinity Φ_t itself converges to zero. Hence, to investigate the relative behaviour of (10) and (11) (in this case) we consider the ratio of the moduli

$$\frac{|\Phi_t - \Phi_t^E|}{|\Phi_t - \Phi_t^D|} = \frac{|\frac{1}{6}\int_0^h e^{s\lambda}s(h-s)(2h-s)z^{(3)}(\theta)\,ds|}{|\varepsilon(t, h, \lambda)/(1 - \frac{1}{2}h\lambda)|}. \tag{13}$$

[4] C.f. [5] and [2].

Keeping the imaginary part of λ constant we take the upper limit of (13) and obtain

$$\limsup_{\mathrm{Re}(\lambda)\to-\infty} \frac{|\Phi_t - \Phi_t^E|}{|\Phi_t - \Phi_t^D|}$$

$$\leqq \frac{\limsup_{\mathrm{Re}(\lambda)\to-\infty} \{|1 - \frac{1}{2}h\lambda|\,|\frac{1}{6}\int_0^h \mathrm{e}^{s\lambda} s(h-s)(2h-s) z^{(3)}(\theta)\,\mathrm{d}s|\}}{\liminf_{\mathrm{Re}(\lambda)\to-\infty} |\varepsilon(t, h, \lambda)|}. \tag{14}$$

But the numerator on the right side of (14) is less than

$$\frac{1}{6}\{\limsup_{\mathrm{Re}(\lambda)\to-\infty} |1-\tfrac{1}{2}h\lambda| \limsup_{\mathrm{Re}(\lambda)\to-\infty} \mathrm{e}^{\alpha \mathrm{Re}(\lambda)}\} \int_0^h s(h-s)(2h-s)|z^{(3)}(\theta)|\,\mathrm{d}s,$$

where $0 < \alpha < h$. Hence, it follows from the fact that $\exp\{-s\,\mathrm{Re}(\lambda)\}$ is a higher-order infinity than $|\mathrm{Re}(\lambda)|$ that the numerator on the right side of (14) is zero. Moreover,

$$\begin{aligned}
&\liminf_{\mathrm{Re}(\lambda)\to-\infty} |\varepsilon(t, h, \lambda)| \\
&\geqq \liminf_{\mathrm{Re}(\lambda)\to-\infty} \left| |(1 - \tfrac{1}{2}h\lambda)\int_0^h \mathrm{e}^{s\lambda} z(th-s)\,\mathrm{d}s| - \tfrac{1}{2}h|z(th) + z(th-h)| \right| \\
&= \liminf_{\mathrm{Re}(\lambda)\to-\infty} \left| \tfrac{1}{2}h|z(th) + z(th-h)| \right. \\
&\qquad \left. - |(1 - \tfrac{1}{2}h\lambda)\int_0^h \mathrm{e}^{s\lambda} z(th-s)\,\mathrm{d}s| \right| \\
&\geqq \tfrac{1}{2}h|z(th) + z(th-h)| \\
&\qquad - \limsup_{\mathrm{Re}(\lambda)\to-\infty} |(1 - \tfrac{1}{2}h\lambda)| \left|\int_0^h \mathrm{e}^{s\lambda} z(th-s)\,\mathrm{d}s\right| \\
&= \tfrac{1}{2}h|z(th) + z(th-h)|,
\end{aligned}$$

which is, in general, greater than zero. Therefore, we have

$$\limsup_{\mathrm{Re}(\lambda)\to-\infty} \frac{|\Phi_t - \Phi_t^E|}{|\Phi_t - \Phi_t^D|} = 0,$$

so that the bias of Φ_t^E tends to zero faster than the bias of Φ_t^D when the damping coefficient $\mathrm{Re}(\lambda)$ becomes infinitely large.

We next consider the case where the imaginary part of λ becomes increasingly large while the real part of λ is taken as fixed. In this case

$$\begin{aligned}
&\limsup_{\mathrm{Im}(\lambda)\to\infty} |\Phi_t - \Phi_t^E| \\
&\leqq \tfrac{1}{6} \limsup_{\mathrm{Im}(\lambda)\to\infty} \int_0^h \mathrm{e}^{s\mathrm{Re}(\lambda)} \left|\mathrm{e}^{s\mathrm{Im}(\lambda)}\right| \left|z^{(3)}(\theta)\right| s(h-s)(2h-s)\,\mathrm{d}s \\
&= \tfrac{1}{6} \int_0^h \mathrm{e}^{s\mathrm{Re}(\lambda)} \left|z^{(3)}(\theta)\right| s(h-s)(2h-s)\,\mathrm{d}s,
\end{aligned}$$

which is bounded uniformly in t. On the other hand,

$$\begin{aligned}
&\liminf_{\mathrm{Im}(\lambda)\to\infty} |\Phi_t - \Phi_t^D| \\
&\geqq \liminf_{\mathrm{Im}(\lambda)\to\infty} \left| \left| \int_0^h \mathrm{e}^{s\lambda} z(th-s)\,\mathrm{d}s \right| - \left| \frac{1}{1-\frac{1}{2}h\lambda} \right| \tfrac{1}{2}h \left| z(th) + z(th-h) \right| \right| \\
&\geqq \liminf_{\mathrm{Im}(\lambda)\to\infty} \left| \int_0^h \mathrm{e}^{s\lambda} z(th-s)\,\mathrm{d}s \right| \\
&\quad - \limsup_{\mathrm{Im}(\lambda)\to\infty} \left\{ \left| \frac{1}{1-\frac{1}{2}h\lambda} \right| \tfrac{1}{2}h \left| z(th) + z(th-h) \right| \right\} \\
&= \liminf_{\mathrm{Im}(\lambda)\to\infty} \left| \int_0^h \mathrm{e}^{s\lambda} z(th-s)\,\mathrm{d}s \right|.
\end{aligned}$$

Hence, we have

$$\limsup_{\mathrm{Im}(\lambda)\to\infty} \frac{|\Phi_t - \Phi_t^E|}{|\Phi_t - \Phi_t^D|} \leqq \frac{\frac{1}{6}\int_0^h \mathrm{e}^{s\mathrm{Re}(\lambda)} \left|z^{(3)}(\theta)\right| s(h-s)(2h-s)\,\mathrm{d}s}{\liminf_{\mathrm{Im}(\lambda)\to\infty} \left|\int_0^h \mathrm{e}^{s\lambda} z(th-s)\,\mathrm{d}s\right|}. \tag{15}$$

Since the numerator on the right side of (15) is not, in general, zero we cannot expect the bias $\Phi_t - \Phi_t^E$ to improve relative to the bias $\Phi_t - \Phi_t^D$ as the frequency [i.e., Im (λ)] of the oscillations generated by the system is allowed to increase indefinitely.

However, it seems reasonable that in many economic models high-frequency components are of much less importance than rapid rates of adjustment. To the extent that high-frequency oscillations do occur in a system the above theory gives us no reason to suppose that the exact model approximation Φ_t^E should deteriorate relative to Φ_t^D. But when some of the equations of a model involve fast adjustment rates, which lead to eigenvalues with large real parts, our theory suggests that the approximation Φ_t^E will improve relative to Φ_t^D even for a fixed sampling interval h.

Thus, from these two observations, it would appear that while (8) may provide a satisfactory approximation for some h and λ it may be unreliable when we have a system with some equations that involve rapid responses. The approximation (9) does not suffer from the same defect and, in addition, has a bias which is smaller in terms of the sampling interval. Hence, if the smoothness properties of z are realistic, (9) would seem to be the superior approximation.

In the following sections of this paper, we will consider whether the results of computations based on observed data series are consistent with this classification of the approximations.

3. Weekly data with the time unit of a quarter

The economic series we consider in this section are:

(i) Financial Times Industrial Share Price Index;
(ii) 91-Day U.K. Treasury Bill Rate;
(iii) U.S. Prime Commercial Paper Rate (4–6 months).

Weekly observations of each of these series were recorded in the years 1965 through to 1972.

To calculate the exogenous variable integral Φ_t and the approximations Φ_t^D and Φ_t^E for integral values of t (representing quarters in the time period under consideration) we first specify the system eigenvalue λ. We do this according to a grid of values for the real and imaginary parts of λ. The real part of λ we classify into three groups:

(a) *Strong damping:* We take 30 values of $\mathrm{Re}(\lambda)$ in the interval $[-3.00, -2.9565]$ according to the scheme,

$$-3.00 + (k - 1)(0.0015), \qquad k = 1, \ldots, 30.$$

When $\mathrm{Re}(\lambda)$ is in this region, the envelope of the system response decays to $1/e$ or 37 % of its initial deviation in approximately 1 month (we say the damping period is 1 month).

(b) *Medium damping:* We take 30 values of Re (λ) in the interval $[-0.25, -0.2065]$ according to the scheme,

$$-0.25 + (k - 1)(0.0015), \qquad k = 1, \ldots, 30.$$

The damping period in this region is approximately 1 year.

(c) *Weak damping:* We take 30 values of Re (λ) in the interval $[-0.05, -0.0065]$ according to the scheme,

$$-0.05 + (k - 1)(0.0015), \qquad k = 1, \ldots, 30.$$

The damping period in this region is between 5 and 39 years.

The imaginary part of λ we also classify into three groups according to the length of the cycle period:[5]

(a) *Short cycle:* Im $(\lambda) = 2.00$. The cycle period is approximately 3 quarters.
(b) *Medium cycle:* Im $(\lambda) = 0.65$. The cycle period is approximately 10 quarters or $2\frac{1}{2}$ years.
(c) *Long cycle:* Im $(\lambda) = 0.20$. The cycle period is approximately 31 quarters or nearly 8 years.

We also consider real eigenvalues and in this case we set up a grid of values of λ that is the same as the one we have just described for the real part of λ when λ is complex.

Using the weekly observations of the series, we calculated the integral defining the exogenous variable component Φ_t by numerical integration for each quarter and for each of the eigenvalues according to the scheme above. Neglecting the intermediate weekly observations, we then used the quarterly observations on the series to calculate the approximations Φ_t^D and Φ_t^E. From the computed values of $\{\Phi_t, \Phi_t^D, \Phi_t^E\colon t = 3, \ldots, T\}$ where $T\,(= 32)$ denotes the total number of quarters, the following statistics were obtained

[5] Although we do not record the results here, 5 groups were actually used in the computations according to the scheme $2.00 + (k - 1)(-0.45)$, $k = 1, \ldots, 5$.

(i) Root error sum of squares (*RESS*):

$$D\text{-model: } RESS = \sqrt{\sum_{t=3}^{T} (\Phi_t - \Phi_t^D)^2};$$

$$E\text{-model: } RESS = \sqrt{\sum_{t=3}^{T} (\Phi_t - \Phi_t^E)^2}.$$

Having found the *RESS* for every eigenvalue in a particular group, we calculated the mean *RESS* in this group for each model. When the system eigenvalue was complex we calculated the RESS separately for the real and imaginary parts.

(ii) Maximum deviation of the approximations from the integral:

$$DMAX = \max_t |\Phi_t - \Phi_t^D|;$$

$$EMAX = \max_t |\Phi_t - \Phi_t^E|.$$

For the eigenvalues in a particular group we recorded the number of times *DMAX* exceeded *EMAS* (denoted by $DMAX > EMAX$). As in (i) when λ was complex the real and imaginary parts were treated separately.

We present these statistics these statistics for the series under consideration in tables 1, 2 and 3. The *RESS* statistic is a measure of the overall performance of the approximations when the system eigenvalue lies in a particular group. We note first that, in the case of each series, the *E*-model has a mean *RESS* which is considerably smaller on the whole than the *D*-model mean *RESS*, when the system eigenvalue has large real and imaginary parts. Tables 1 and 2 indicate also that, even for systems moderate damping factors and medium cycles, the *E*-model approximations seems to be superior according to this criterion. For eigenvalues in this latter class, however, the results of table 3 are a little different. We notice here that the approximations are very close according to this criterion. For eigenvalues with long cycles and weak damping factors all tables suggest that there is little difference between the approximations.

We also record in the tables the number of times $DMAX > EMAX$. The measures *DMAX* and *EMAX* indicate the worst performance of

Table 1
Financial times share price index.

		Strong damping $(-3.00, -2.9565)$[a]	Medium damping $(-0.25, -0.2065)$[a]	Weak damping $(-0.05, -0.0065)$[a]
Real eigenvalue	E-model: Mean $RESS$	6.7536	18.7002	20.6694
	D-model: Mean $RESS$	66.6818	18.4423	20.6258
	$DMAX > EMAX$	30	0	5
Complex eigenvalue	*Real parts*			
	E-model: Mean $RESS$	5.1842	10.8725	11.5931
	D-model: Mean $RESS$	83.0232	53.0481	39.9758
Short cycle	$DMAX > EMAX$	30	30	30
$(\mathrm{Im}\,\lambda = 2.00)$	*Imaginary parts*			
	E-model: Mean $RESS$	3.6093	13.3882	14.9376
	D-model: Mean $RESS$	8.3925	144.6965	172.4440
	$DMAX > EMAX$	30	30	30
	Real parts			
	E-model: Mean $RESS$	6.5672	17.6993	19.3524
	D-model: Mean $RESS$	68.5976	21.1501	31.4712
Medium cycle	$DMAX > EMAX$	30	30	30
$(\mathrm{Im}\,\lambda = 0.65)$	*Imaginary parts*			
	E-model: Mean $RESS$	1.3409	5.4936	6.1882
	D-model: Mean $RESS$	4.9632	25.6309	19.9950
	$DMAX > EMAX$	30	30	30
	Real parts			
	E-model: Mean $RESS$	6.7357	18.6035	20.3904
	D-model: Mean $RESS$	66.8647	18.5318	21.2582
Long cycle	$DMAX > EMAX$	30	0	30
$(\mathrm{Im}\,\lambda = 0.20)$	*Imaginary parts*			
	E-model: Mean $RESS$	0.4186	1.7334	1.9547
	D-model: Mean $RESS$	1.6020	5.4608	2.6970
	$DMAX > EMAX$	30	30	30

[a] We consider 30 different values of $\mathrm{Re}(\lambda)$ on a grid within this interval.

Table 2
U.K. treasury bill rate.

		Strong damping $(-3.00, -2.9565)$[a]	Medium damping $(-0.25, -0.2065)$[a]	Weak damping $(-0.05, -0.0065)$[a]
Real eigenvalue	*E*-model: Mean *RESS*	0.2383	0.8083	0.8999
	D-model: Mean *RESS*	2.8793	0.8085	0.8747
	DMAX > *EMAX*	30	0	0
Complex eigenvalue	*Real parts*			
	E-model: Mean *RESS*	0.1743	0.4028	0.4332
	D-model: Mean *RESS*	3.5304	2.3358	1.7873
Short cycle	*DMAX* > *EMAX*	30	30	30
($\mathrm{Im}\,\lambda = 2.00$)	*Imaginary parts*			
	E-model: Mean *RESS*	0.1656	0.6580	0.7380
	D-model: Mean *RESS*	0.2948	5.7848	6.9113
	DMAX > *EMAX*	30	30	30
	Real parts			
	E-model: Mean *RESS*	0.2302	0.7567	0.8389
	D-model: Mean *RESS*	2.9556	0.7222	1.0382
Medium cycle	*DMAX* > *EMAX*	30	11	30
($\mathrm{Im}\,\lambda = 0.65$)	*Imaginary parts*			
	E-model: Mean *RESS*	0.0629	0.2726	0.3075
	D-model: Mean *RESS*	0.1848	0.9696	0.7420
	DMAX > *EMAX*	30	30	30
	Real parts			
	E-model: Mean *RESS*	0.2375	0.8033	0.8923
	D-model: Mean *RESS*	2.8866	0.7850	0.8497
Long cycle	*DMAX* > *EMAX*	30	0	0
($\mathrm{Im}\,\lambda = 0.20$)	*Imaginary parts*			
	E-model: Mean *RESS*	0.0916	0.0860	0.0971
	D-model: Mean *RESS*	0.0598	0.2020	0.1256
	DMAX > *EMAX*	30	30	19

[a] We consider 30 different values of $\mathrm{Re}(\lambda)$ on a grid within this interval.

Table 3
U.S. prime commercial paper rate.

		Strong damping $(-3.00, -2.9565)$[a]	Medium damping $(-0.25, -0.2065)$[a]	Weak damping $(-0.05, -0.0065)$[a]
Real eigenvalue	E-model: Mean $RESS$	1.2304	4.6174	5.1753
	D-model: Mean $RESS$	2.7572	3.9975	4.4995
	$DMAX > EMAX$	0	0	0
Complex eigenvalue	*Real parts*			
	E-model: Mean $RESS$	0.8029	1.7897	1.8978
	D-model: Mean $RESS$	3.2720	2.5190	2.1860
Short cycle	$DMAX > EMAX$	30	0	0
($\mathrm{Im}\,\lambda = 2.00$)	*Imaginary parts*			
	E-model: Mean $RESS$	0.7882	3.7262	4.2417
	D-model: Mean $RESS$	0.7533	6.5460	7.7176
	$DMAX > EMAX$	0	0	0
	Real parts			
	E-model: Mean $RESS$	1.1778	4.2435	4.7501
	D-model: Mean $RESS$	2.8118	3.7166	4.2595
Medium cycle	$DMAX > EMAX$	0	0	0
($\mathrm{Im}\,\lambda = 0.65$)	*Imaginary parts*			
	E-model: Mean $RESS$	0.3130	1.6805	1.9306
	D-model: Mean $RESS$	0.3346	1.7960	1.8684
	$DMAX > EMAX$	0	0	0
	Real parts			
	E-model: Mean $RESS$	0.7882	4.5812	5.1447
	D-model: Mean $RESS$	0.7533	3.9695	4.4723
Long cycle	$DMAX > EMAX$	0	0	0
($\mathrm{Im}\,\lambda = 0.20$)	*Imaginary parts*			
	E-model: Mean $RESS$	0.0984	0.5348	0.6149
	D-model: Mean $RESS$	9.1062	0.5162	0.5413
	$DMAX > EMAX$	0	0	0

[a] We consider 30 different values of $\mathrm{Re}(\lambda)$ on a grid within this interval.

the D-model and E-model approximations over all quarters for which the approximations were computed. Thus, if an approximation performs poorly for one or two quarters but well on the whole then its poor performance will figure in this maximum deviation measure, whereas it will be relatively less important in the mean *RESS* statistic.

Tables 1 and 2 show us that, for the first two series, the maximum deviation measure gives results which are consistent with those of the mean *RESS* criterion. For both series we can conclude is by far the more reliable over a range of different eigenvalues and, when the system eigenvalue is large, it provides an approximation which is much better than that of the D-model. These conclusions are supported by an inspection of the figures in charts 1 and 2 where we graph the approximations and the exogenous variable component for a representative selection of eigenvalues.[6] In both charts we see that Φ_t^D is appreciable more biased than Φ_t^E when the system eigenvalue has a large modulus. On the other hand, the approximations are very close when the system eigenvalue has a small real part and a small imaginary part.

Turning now to the third series we notice from table 3 that according to the maximum deviation measure Φ_t^E performs rather worse on the whole than Φ_t^D. Looking at the figures in chart 3 we see the reason for this: it is clear from these figures that for one particular quarter (the 24th) Φ_t^E seriously underestimates the exogenous variable component, whereas for the remaining quarters its performance is satisfactory. Intuitively, we might expect that the movement of the series is irregular in a region of the 24th quarter (i.e., around the 312 consecutive observation of the series) and this is confirmed by an inspection of figure 3 in the appendix, where we have graphed the series.

As mentioned in the introduction to this paper, the theory developed in [2] suggests that the performance of the E-model approximation depends on the smoothness property of the exogenous series. One way of measuring this property is to regress the series on a polynomial of time and use the coefficient of determination, R^2, from this regression as an indicator of the smoothness of the series. We would expect a smooth series to have a higher R^2 in a regression of this type than an irregular series.

[6] A random integer between 1 and 30 was chosen and according to this number we selected the eigenvalue whose real part corresponded to this point on the grid in the strong damping and weak damping groups.

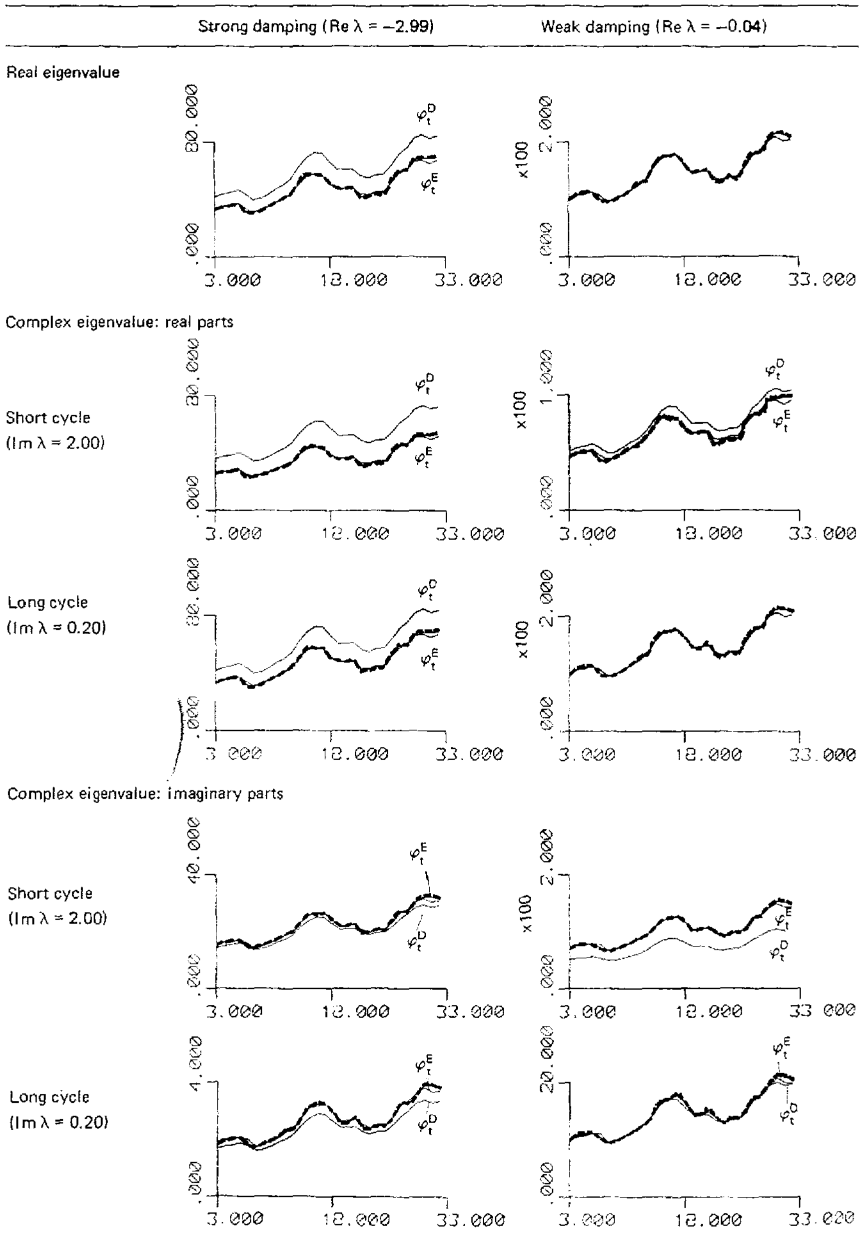


Chart 1. Financial times industrial share price index. Number of quarters is measured on the x-axis, ---- denotes Φ_t.

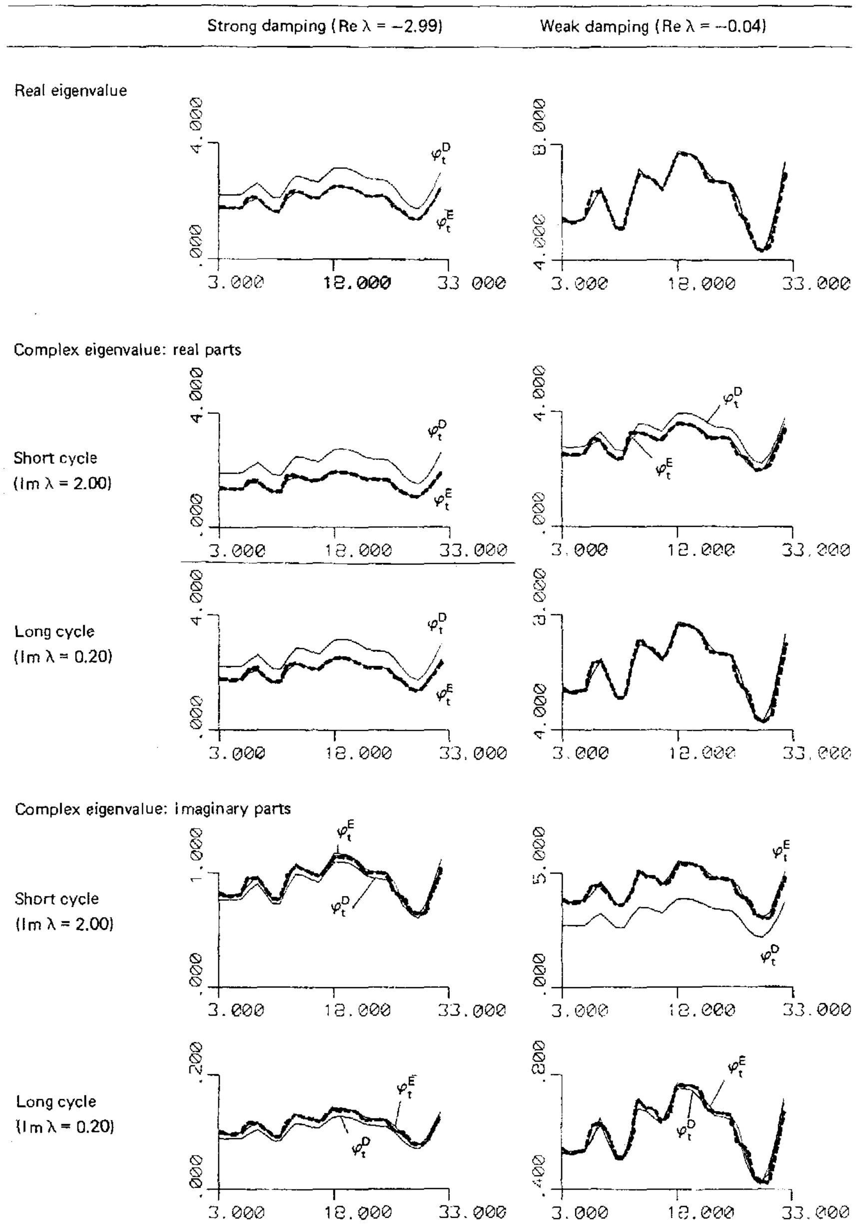


Chart 2. U.K. treasury bill rate. Number of quarters is measured on the x-axis, ---- denotes Φ_t.

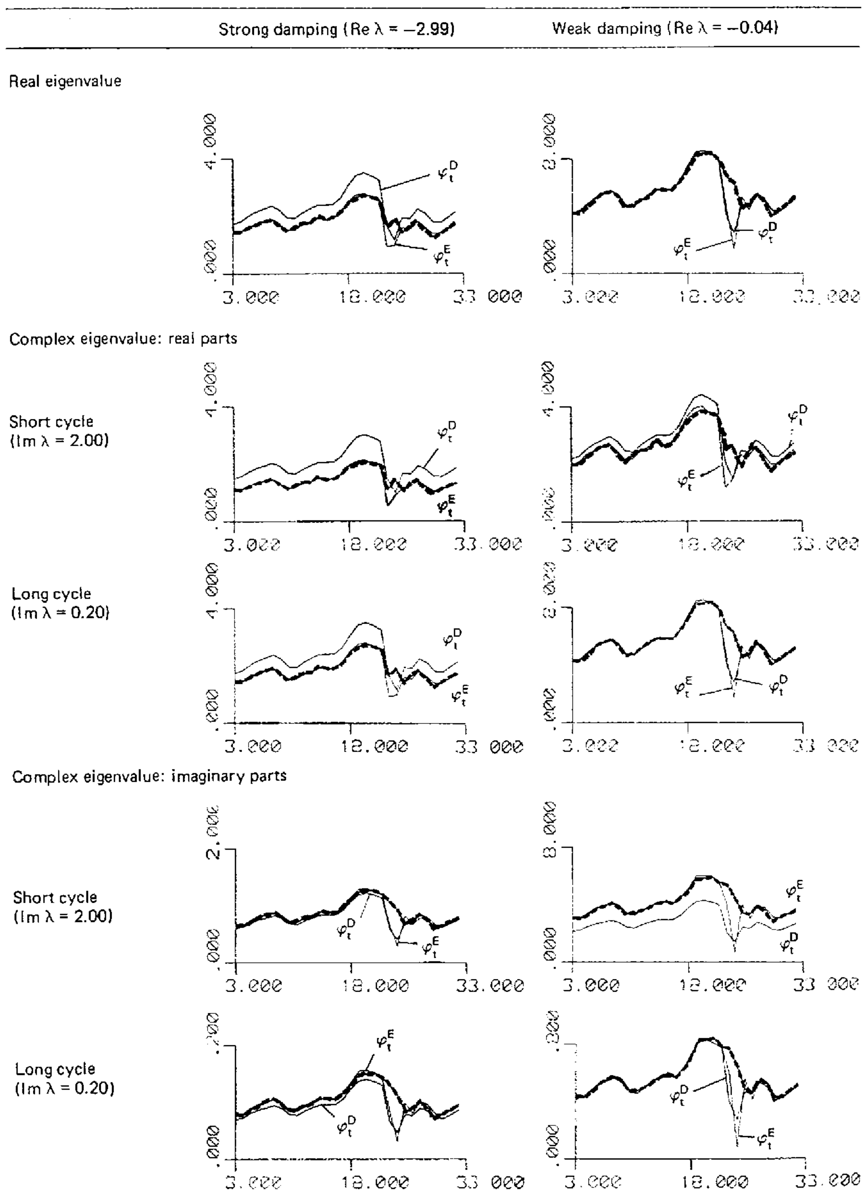


Chart 3. U.S. prime commercial paper rate. Number of quarters is measured on the x-axis, ---- denotes Φ_t.

Using the quarterly observations we computed the R^2 in the regression

$$y_t = a_0 + a_1 t + a_2 t^2 + \ldots + a_6 t^6 + u_t, \tag{16}$$

for each series and we detail the results below:

Series	Smoothness coefficient (R^2)
(i) Financial Times Share Index	0.9226
(ii) Treasury Bill Rate	0.8099
(iii) U.S. Prime Commercial Paper Rate	0.7457

According to the stated criterion the third series displays more irregularity than the others and this gives us a meaningful explanation of the observed result that the E-model approximation does not perform quite as well for this series as the others.

4. *Monthly data with the time unit of a year*

In this section we consider the following series, for which monthly observations were recorded in the years indicated:

(iv) U.K. Index of Industrial Production: 1947–1971.
(v) U.K. Registered Unemployed: 1947–1971.
(vi) U.K. Import Price Index (Food): 1947–1971.
(vii) U.K. Import Price Index (Total): 1947–1971.
(viii) World Commodity Price Index (Metals): 1949–1972.
(ix) U.K. Exports (Visible Trade): 1947–1971.
(x) U.K. Imports (Visible Trade): 1947–1971.

As in section 3 we first construct a grid for the system eigenvalue λ. Since the time unit is now a year we can expect the range of realistic eigenvalues to be somewhat greater. Thus, for the real part of λ we specify the following groups:

(a) *Strong damping:* We take 30 values of Re(λ) in the interval $[-4.50, -4.21]$ according to the scheme,

$$-4.50 + (k-1)(0.01), \qquad k = 1, \ldots, 30.$$

The damping period in this region is approximately $2\frac{1}{2}$ months.

(b) *Medium damping:* We take 30 values of $\mathrm{Re}(\lambda)$ in the interval $[-2.00, -1.71]$ according to the scheme,

$$-2.00 + (k-1)(0.01), \qquad k = 1, \ldots, 30.$$

The damping period in this region is approximately 1 year.

(c) *Weak damping:* We take 30 values of $\mathrm{Re}(\lambda)$ in the interval $[-0.30, -0.01]$ according to the scheme,

$$-0.30 + (k-1)(0.01), \qquad k = 1, \ldots, 30.$$

The damping period in this region is between 3 and 100 years.

We specify the following groups[7] for the imaginary part of λ:

(a) *Short cycle:* $\mathrm{Im}(\lambda) = 3.00$. The cycle period is approximately 2 years.
(b) *Medium cycle:* $\mathrm{Im}(\lambda) = 0.80$. The cycle period is approximately 8 years.
(c) *Long cycle:* $\mathrm{Im}(\lambda) = 0.25$. The cycle period is approximately 25 years.

Real eigenvalues were considered also and these were classified into the same groups as those for the real parts in the complex case above.

As in section 3, we computed the exogenous variable integral Φ_t by numerical integration taking account of the intermediate montly observations, but used only annual observations to compute the approximations Φ_t^D and Φ_t^E. A methodological problem arises in the treatment of series (ix) and (x) because instantaneous observations of exports and imports are not available. In another paper [4], we have shown that when we have a flow variable model the exact discrete model can be integrated over an appropriate time interval and estimated with flow data in the resulting form. If we carry out this procedure, then the exogenous variable component (3) becomes

$$\int_0^h e^{s\lambda} \left\{ \int_{th-h-s}^{th-s} z(\tau)\, d\tau \right\} ds. \tag{17}$$

Instead of calculating (3), therefore, we calculate (17). We can do this numerically because we have monthly observations of the series and by

[7] Although we do not record the results here, 6 groups were actually used in the computations according to the scheme $3.00 + (k-1)(-0.55)$, $k = 1, \ldots, 6$.

temporal aggregation we can obtain intermediate observations of the quantity in braces in the integrand of (17). The approximations Φ_t^D and Φ_t^E are then approximations to (17) and are computed from annual totals of the series.

For each series (iv) to (x) we calculated the summary statistics described in section 3 and these are presented in tables 4–10. We see from these tables that the E-model approximation performs considerably better in terms of the mean *RESS* statistic than the D-model approximation when the eigenvalue λ has large real and imaginary parts. This result accords well with what we have observed for the weekly series, and it is supported by an inspection of charts 4–10 where we graph the integral Φ_t and the approximations Φ_t^D and Φ_t^E for a random selection of eigenvalues (obtained in the same way[8] as in section 3). We notice in these charts that when we have strong damping and short cycles Φ_t^D exhibits uniformly more bias than Φ_t^E.

For the case of medium damping and medium cycles we see from tables 4–10 that the E-model approximation still has a mean *RESS* which is much smaller on the whole than that of the D-model approximation. The only series for which the D-model approximation comes close to performing as well as the E-model approximation for eigenvalues in this category is the World Metals Price Index [series (viii), table 8].

When the system eigenvalues have small real and imaginary parts we observe in all tables that the mean *RESS* for the different approximations are quite close. The figures in charts 4–10 bear out this result and we notice that for the weak damping, long-cycle category the graphs of Φ_t, Φ_t^D and Φ_t^E are frequently so close that they are difficult to distinguish.

In tables 4–10 we record also the maximum deviation statistic. For all series but the Metals Index (table 8), the E-model approximation scores consistently better than the D-model approximation according to this criterion. In the corner of table 8 corresponding to medium-long cycles and medium-weak damping we notice that *EMAX* is never exceeded by *DMAX*. This result suggests that the E-model approximation may be performing badly for some years. An inspection of the figures in chart 8 supports this conjecture. In the figures, we see that Φ_t^D displays

[8] C.f. footnote 6.

Table 4
U.K. index of industrial production.

		Strong damping $(-4.50, -4.21)$[a]	Medium damping $(-2.0, -1.71)$[a]	Weak damping $(-0.30, -0.01)$[a]
Real eigenvalue	*E*-model: Mean *RESS*	3.3602	5.1825	9.5868
	D-model: Mean *RESS*	43.8746	30.5958	8.9658
	$DMAX > EMAX$	30	30	8
Complex eigenvalue	*Real parts*			
	E-model: Mean *RESS*	3.1176	2.9581	2.3308
	D-model: Mean *RESS*	49.9437	74.8334	129.3598
Short cycle	$DMAX > EMAX$	30	30	30
(Im $\lambda = 3.00$)	*Imaginary parts*			
	E-model: Mean *RESS*	1.1270	2.6917	5.7948
	D-model: Mean *RESS*	6.6716	13.3827	94.1571
	$DMAX > EMAX$	30	30	30
	Real parts			
	E-model: Mean *RESS*	3.3271	4.9229	8.6634
	D-model: Mean *RESS*	44.5453	34.0222	17.0433
Medium cycle	$DMAX > EMAX$	30	30	30
(Im $\lambda = 0.80$)	*Imaginary parts*			
	E-model: Mean *RESS*	0.3833	1.1656	3.4170
	D-model: Mean *RESS*	0.4409	11.3456	22.3777
	$DMAX > EMAX$	21	30	30
	Real parts			
	E-model: Mean *RESS*	3.3568	5.1564	9.5327
	D-model: Mean *RESS*	43.9418	30.9217	9.8626
Long cycle	$DMAX > EMAX$	30	30	21
(Im $\lambda = 0.25$)	*Imaginary parts*			
	E-model: Mean *RESS*	0.1219	0.3781	1.1267
	D-model: Mean *RESS*	0.1296	3.6355	3.5653
	$DMAX > EMAX$	18	30	30

[a] We consider 30 different values of $\text{Re}(\lambda)$ on a grid within this level.

Table 5
U.K. registered unemployed.

		Strong damping $(-4.50, -4.21)$[a]	Medium damping $(-2.0, -1.71)$[a]	Mean damping $(-0.30, -0.01)$[a]
Real eigenvalue	*E*-model: Mean *RESS*	31.4735	70.0437	139.4822
	D-model: Mean *RESS*	210.8296	179.6182	144.9223
	$DMAX > EMAX$	30	30	30
Complex eigenvalue	*Real parts*			
	E-model: Mean *RESS*	19.0922	40.2584	94.6732
	D-model: Mean *RESS*	229.9480	350.1055	614.4609
Short cycle ($\mathrm{Im}\,\lambda = 3.00$)	$DMAX > EMAX$	30	30	30
	Imaginary parts			
	E-model: Mean *RESS*	22.6664	56.6346	116.4824
	D-model: Mean *RESS*	45.1895	59.0811	371.5620
	$DMAX > EMAX$	30	30	30
	Real parts			
	E-model: Mean *RESS*	30.4665	66.9218	128.4884
	D-model: Mean *RESS*	213.1595	192.2370	130.1988
Medium cycle ($\mathrm{Im}\,\lambda = 0.80$)	$DMAX > EMAX$	30	30	30
	Imaginary parts			
	E-model: Mean *RESS*	7.3903	20.8626	56.9987
	D-model: Mean *RESS*	8.1049	47.0396	115.3362
	$DMAX > EMAX$	30	30	30
	Real parts			
	E-model: Mean *RESS*	31.3742	69.7281	138.2201
	D-model: Mean *RESS*	211.0642	180.8008	142.3333
Long cycle ($\mathrm{Im}\,\lambda = 0.25$)	$DMAX > EMAX$	30	30	30
	Imaginary parts			
	E-model: Mean *RESS*	2.3425	6.7052	18.7444
	D-model: Mean *RESS*	2.4749	15.2005	27.6543
	$DMAX > EMAX$	28	30	30

[a] We consider 30 different values of $\mathrm{Re}(\lambda)$ on a grid within this level.

Table 6
U.K. import price index (food).

		Strong damping $(-4.50, -4.21)$[a]	Medium damping $(-2.0, -1.71)$[a]	Weak damping $(-0.30, -0.01)$[a]
Real eigenvalue	*E*-model: Mean *RESS*	1.6718	3.8152	7.8575
	D-model: Mean *RESS*	46.6510	34.7591	7.5877
	DMAX > *EMAX*	30	30	0
Complex eigenvalue	*Real parts*			
	E-model: Mean *RESS*	1.0538	1.6836	2.4940
	D-model: Mean *RESS*	52.4985	76.9075	128.5133
Short cycle ($\mathrm{Im}\,\lambda = 3.00$)	*DMAX* > *EMAX*	30	30	30
	Imaginary parts			
	E-model: Mean *RESS*	1.1511	2.9755	6.3688
	D-model: Mean *RESS*	7.2914	12.2386	90.2041
	DMAX > *EMAX*	30	30	30
	Real parts			
	E-model: Mean *RESS*	1.6166	3.6028	7.1670
	D-model: Mean *RESS*	47.2929	37.9681	10.6853
Medium cycle ($\mathrm{Im}\,\lambda = 0.80$)	*DMAX* > *EMAX*	30	30	26
	Imaginary parts			
	E-model: Mean *RESS*	0.3929	1.1610	2.8341
	D-model: Mean *RESS*	0.5647	10.4861	19.5926
	DMAX > *EMAX*	23	30	30
	Real parts			
	E-model: Mean *RESS*	1.6664	3.7940	7.7168
	D-model: Mean *RESS*	46.7153	35.0635	6.9928
Long cycle ($\mathrm{Im}\,\lambda = 0.25$)	*DMAX* > *EMAX*	30	30	0
	Imaginary parts			
	E-model: Mean *RESS*	0.1249	0.3726	0.1973
	D-model: Mean *RESS*	0.1525	3.3528	2.7373
	DMAX > *EMAX*	16	30	30

[a] We consider 30 different values of $\mathrm{Re}(\lambda)$ on a grid within this level.

Table 7
U.K. import price index (total).

		Strong damping $(-4.50, -4.21)$[a]	Medium damping $(-2.0, -1.71)$[a]	Weak damping $(-0.30, -0.01)$[a]
Real eigenvalue	*E*-model: Mean *RESS*	1.7267	4.3067	9.4568
	D-model: Mean *RESS*	45.4927	33.1637	10.1601
	DMAX > *EMAX*	30	30	1
Complex eigenvalue	*Real parts*			
	E-model: Mean *RESS*	0.9674	1.5271	2.3517
	D-model: Mean *RESS*	51.4840	75.4997	126.6792
Short cycle	*DMAX* > *EMAX*	30	30	30
(Im λ = 3.00)	*Imaginary parts*			
	E-model: Mean *RESS*	1.2326	3.4581	7.8222
	D-model: Mean *RESS*	6.9101	13.2253	91.1456
	DMAX > *EMAX*	30	30	30
	Real parts			
	E-model: Mean *RESS*	1.6570	4.0187	8.5673
	D-model: Mean *RESS*	46.1451	36.3863	14.3055
Medium cycle	*DMAX* > *EMAX*	30	30	30
(Im λ = 0.80)	*Imaginary parts*			
	E-model: Mean *RESS*	0.4408	1.4340	3.6974
	D-model: Mean *RESS*	0.5809	10.8168	20.8183
	DMAX > *EMAX*	19	30	30
	Real parts			
	E-model: Mean *RESS*	1.7198	4.2779	9.3399
	D-model: Mean *RESS*	45.5581	33.4694	10.1661
Long cycle	*DMAX* > *EMAX*	30	30	29
(Im λ = 0.25)	*Imaginary parts*			
	E-model: Mean *RESS*	0.1406	0.4618	1.2000
	D-model: Mean *RESS*	0.1692	3.4612	3.3026
	DMAX > *EMAX*	26	30	24

[a] We consider 30 different values of Re(λ) on a grid within this level.

Table 8
World commodity price index (metals).

		Strong damping $(-4.50, -4.21)$[a]	Medium damping $(-2.0, -1.71)$[a]	Weak damping $(-0.30, -0.01)$[a]
Real eigenvalue	*E*-model: Mean *RESS*	20.5891	44.1195	96.8443
	D-model: Mean *RESS*	62.7613	57.3905	87.1035
	DMAX > *EMAX*	30	0	0
Complex eigenvalue	*Real parts*			
	E-model: Mean *RESS*	14.5449	19.8480	31.2308
	D-model: Mean *RESS*	69.7578	103.2004	178.7844
Short cycle	*DMAX* > *EMAX*	30	30	30
($\mathrm{Im}\,\lambda = 3.00$)	*Imaginary parts*			
	E-model: Mean *RESS*	11.3813	30.1660	69.8845
	D-model: Mean *RESS*	13.5458	33.2895	142.4403
	DMAX > *EMAX*	0	0	0
	Real parts			
	E-model: Mean *RESS*	20.0125	41.1894	86.2193
	D-model: Mean *RESS*	63.5026	59.2323	80.9680
Medium cycle	*DMAX* > *EMAX*	30	0	0
($\mathrm{Im}\,\lambda = 0.80$)	*Imaginary parts*			
	E-model: Mean *RESS*	4.0693	13.9513	40.5606
	D-model: Mean *RESS*	3.7178	20.0216	48.1964
	DMAX > *EMAX*	0	0	0
	Real parts			
	E-model: Mean *RESS*	20.5316	43.8240	95.7975
	D-model: Mean *RESS*	62.8352	57.5378	86.4539
Long cycle	*DMAX* > *EMAX*	30	0	0
($\mathrm{Im}\,\lambda = 0.25$)	*Imaginary parts*			
	E-model: Mean *RESS*	1.2994	4.5289	13.3228
	D-model: Mean *RESS*	1.1856	6.4502	13.0952
	DMAX > *EMAX*	0	0	0

[a] We consider 30 different values of $\mathrm{Re}(\lambda)$ on a grid within this level.

Table 9
U.K. exports (visible trade).

		Strong damping $(-4.50, -4.21)$[a]	Medium damping $(-2.0, -1.71)$[a]	Weak damping $(-0.30, -0.01)$[a]
Real eigenvalue	*E*-model: Mean *RESS*	26.1841	75.0119	187.2422
	D-model: Mean *RESS*	1714.9081	1225.2979	252.6817
	DMAX > *EMAX*	30	30	29
Complex eigenvalue	*Real parts*			
	E-model: Mean *RESS*	14.1084	25.0541	54.2448
	D-model: Mean *RESS*	1950.7301	2841.1748	4725.2910
Short cycle ($\text{Im}\,\lambda = 3.00$)	*DMAX* > *EMAX*	30	30	30
	Imaginary parts			
	E-model: Mean *RESS*	20.9369	63.0640	153.0444
	D-model: Mean *RESS*	242.4676	524.8455	3555.4767
	DMAX > *EMAX*	30	30	30
	Real parts			
	E-model: Mean *RESS*	24.9075	68.6839	163.2790
	D-model: Mean *RESS*	1740.2047	1345.4388	465.9574
Medium cycle ($\text{Im}\,\lambda = 0.80$)	*DMAX* > *EMAX*	16	30	30
	Imaginary parts			
	E-model: Mean *RESS*	7.8721	28.7813	81.2628
	D-model: Mean *RESS*	11.8215	409.3446	711.6842
	DMAX > *EMAX*	19	30	30
	Real parts			
	E-model: Mean *RESS*	26.0564	74.3767	181.9447
	D-model: Mean *RESS*	1717.4397	1236.6504	240.1208
Long cycle ($\text{Im}\,\lambda = 0.25$)	*DMAX* > *EMAX*	30	30	30
	Imaginary parts			
	E-model: Mean *RESS*	2.5206	9.3238	26.5308
	D-model: Mean *RESS*	3.2929	130.4510	98.9384
	DMAX > *EMAX*	20	30	30

[a] We consider 30 different values of $\text{Re}(\lambda)$ on a grid within this level.

Table 10
U.K. imports (visible trade).

		Strong damping $(-4.50, -4.21)$[a]	Medium damping $(-2.0, 1.71)$[a]	Weak damping $(-0.30, -0.01)$[a]
Real eigenvalue	*E*-model: Mean *RESS*	59.4682	134.0020	281.1261
	D-model: Mean *RESS*	2101.7673	1501.0155	264.3409
	DMAX > *EMAX*	30	30	30
Complex eigenvalue	*Real parts*			
	E-model: Mean *RESS*	36.6675	50.7590	56.6165
	D-model: Mean *RESS*	2390.0947	3487.5868	5818.1279
Short cycle	*DMAX* > *EMAX*	30	30	30
($\mathrm{Im}\,\lambda = 3.00$)	*Imaginary parts*			
	E-model: Mean *RESS*	36.9087	97.2829	215.3931
	D-model: Mean *RESS*	300.6781	642.5283	4341.9743
	DMAX > *EMAX*	30	30	30
	Real parts			
	E-model: Mean *RESS*	57.4526	125.8116	254.0993
	D-model: Mean *RESS*	2132.7840	1650.4215	591.3586
Medium cycle	*DMAX* > *EMAX*	30	30	30
($\mathrm{Im}\,\lambda = 0.80$)	*Imaginary parts*			
	E-model: Mean *RESS*	12.9398	40.3543	103.3270
	D-model: Mean *RESS*	17.3861	504.6924	902.1277
	DMAX > *EMAX*	30	30	30
	Real parts			
	E-model: Mean *RESS*	59.2683	133.1850	276.1611
	D-model: Mean *RESS*	2104.8719	1515.1696	266.1808
Long cycle	*DMAX* > *EMAX*	30	30	30
($\mathrm{Im}\,\lambda = 0.25$)	*Imaginary parts*			
	E-model: Mean *RESS*	4.1229	12.9972	33.5554
	D-model: Mean *RESS*	4.9781	161.0084	125.8207
	DMAX > *EMAX*	30	30	30

[a] We consider 30 different values of $\mathrm{Re}(\lambda)$ on a grid within this level.

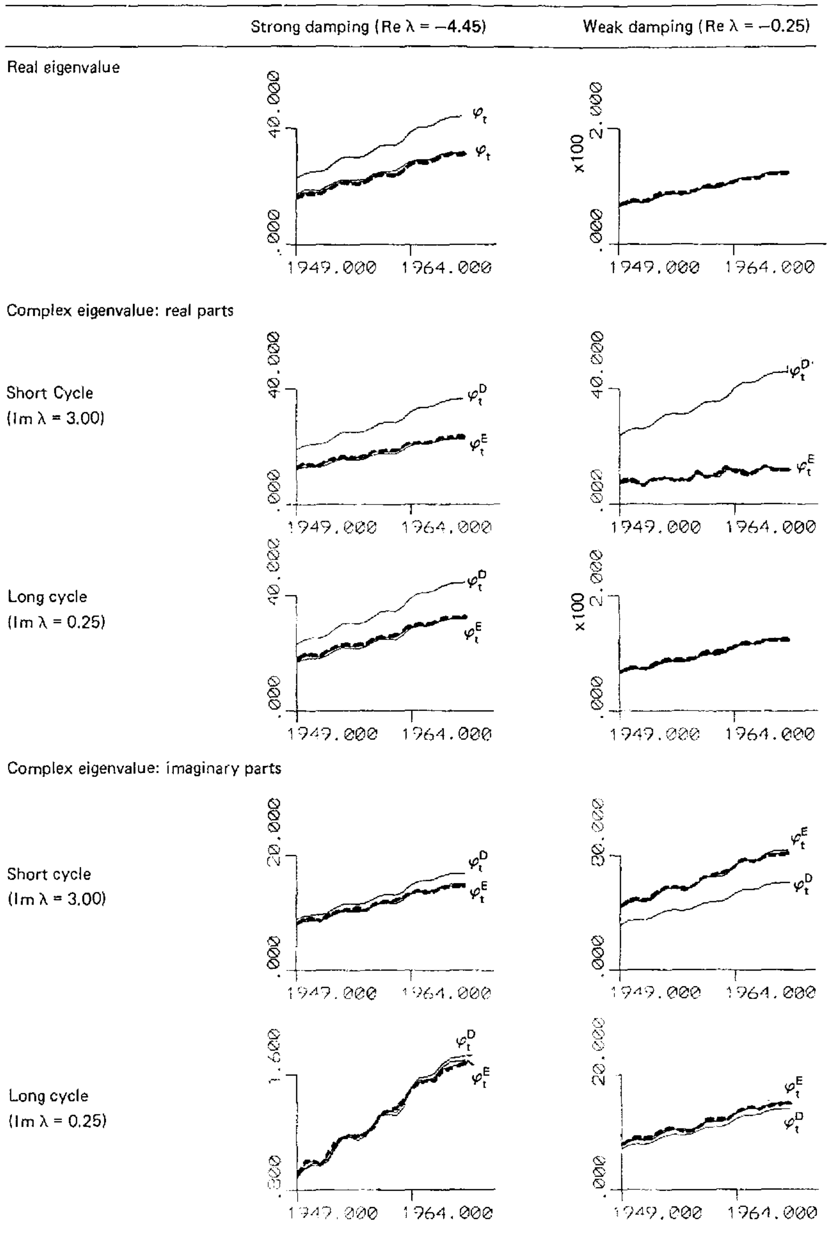


Chart 4. U.K. index of industrial production. ——— denotes Φ_t.

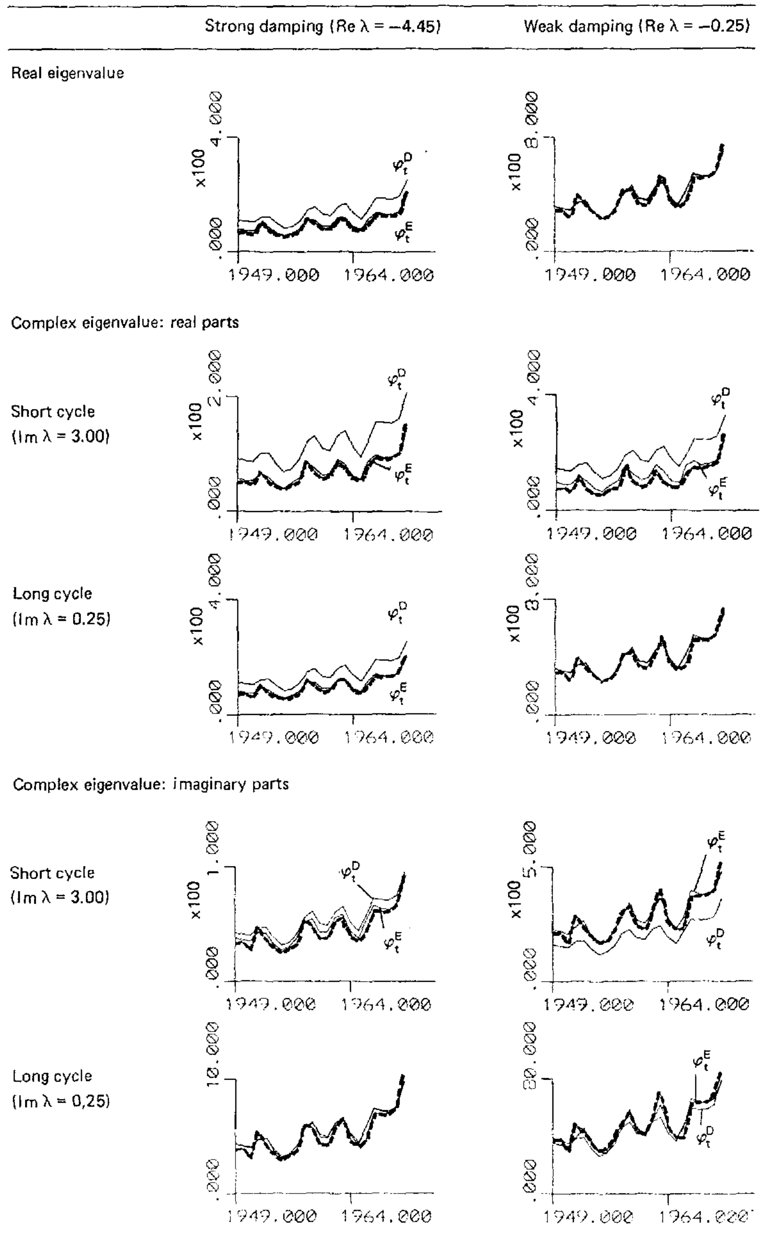


Chart 5. U.K. registered unemployed. ---- denotes Φ_t.

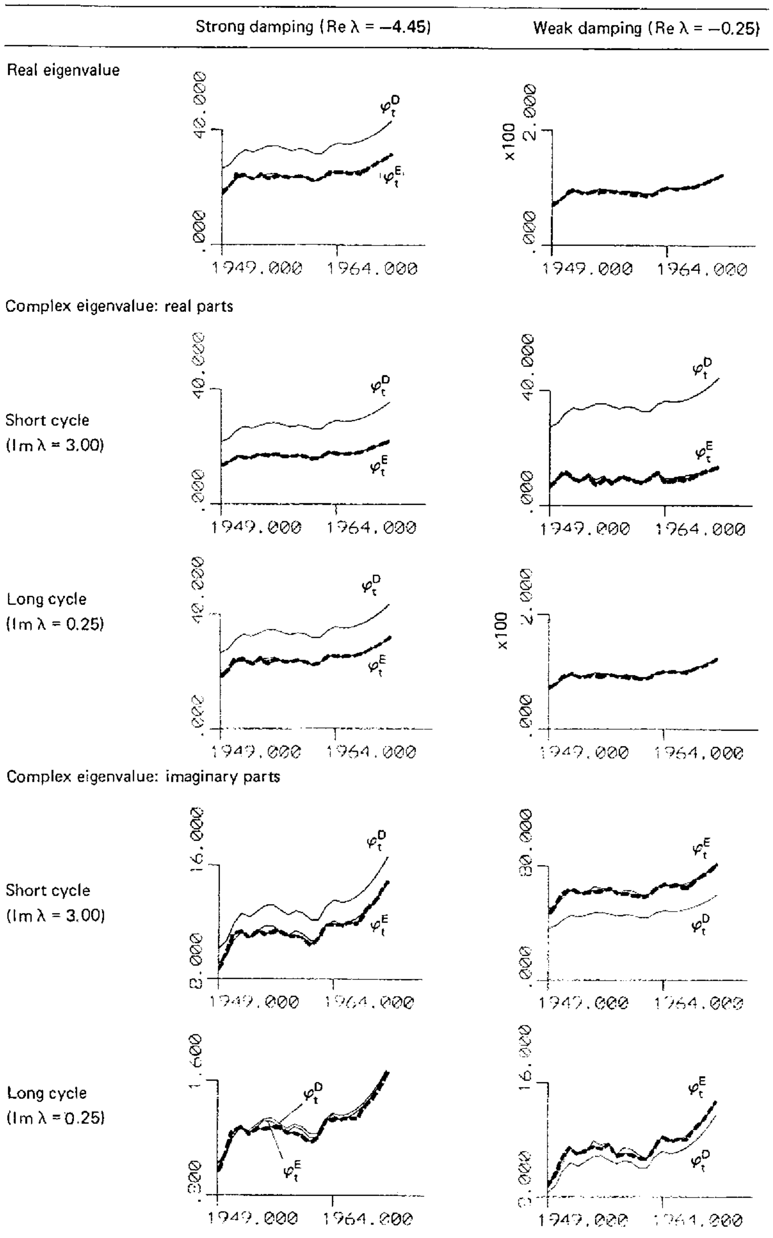


Chart 6. U.K. import price index (food). ––––– denotes Φ_t.

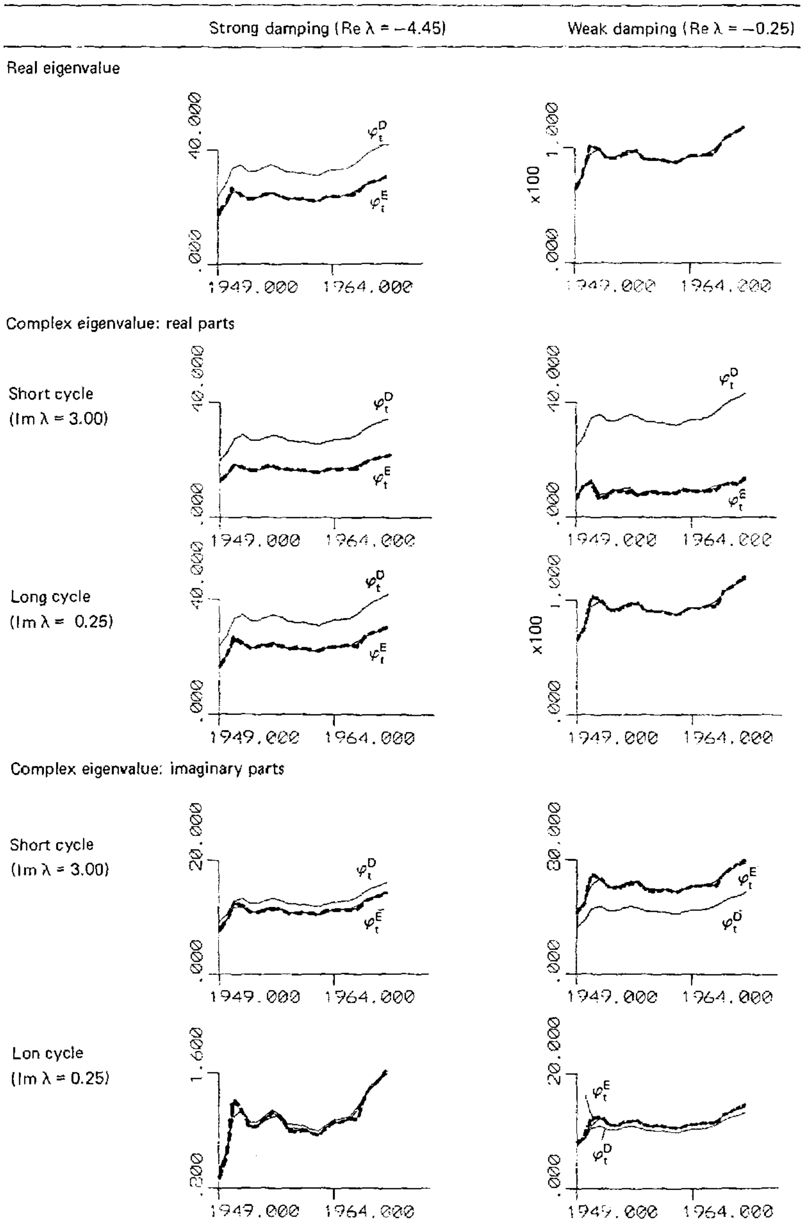


Chart 7. U.K. import price index (total). ---- denotes Φ_t.

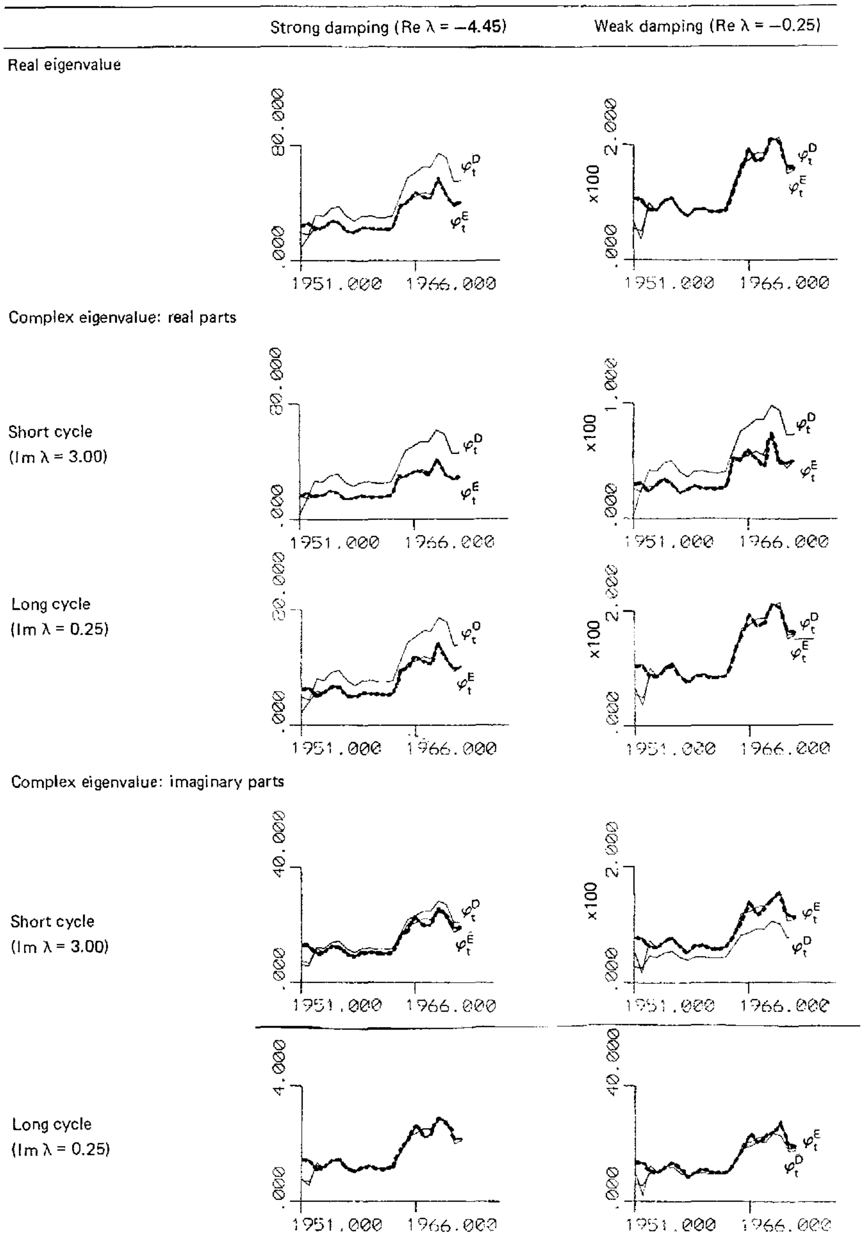


Chart 8. World commodity price index (metals). ----- denotes Φ_t.

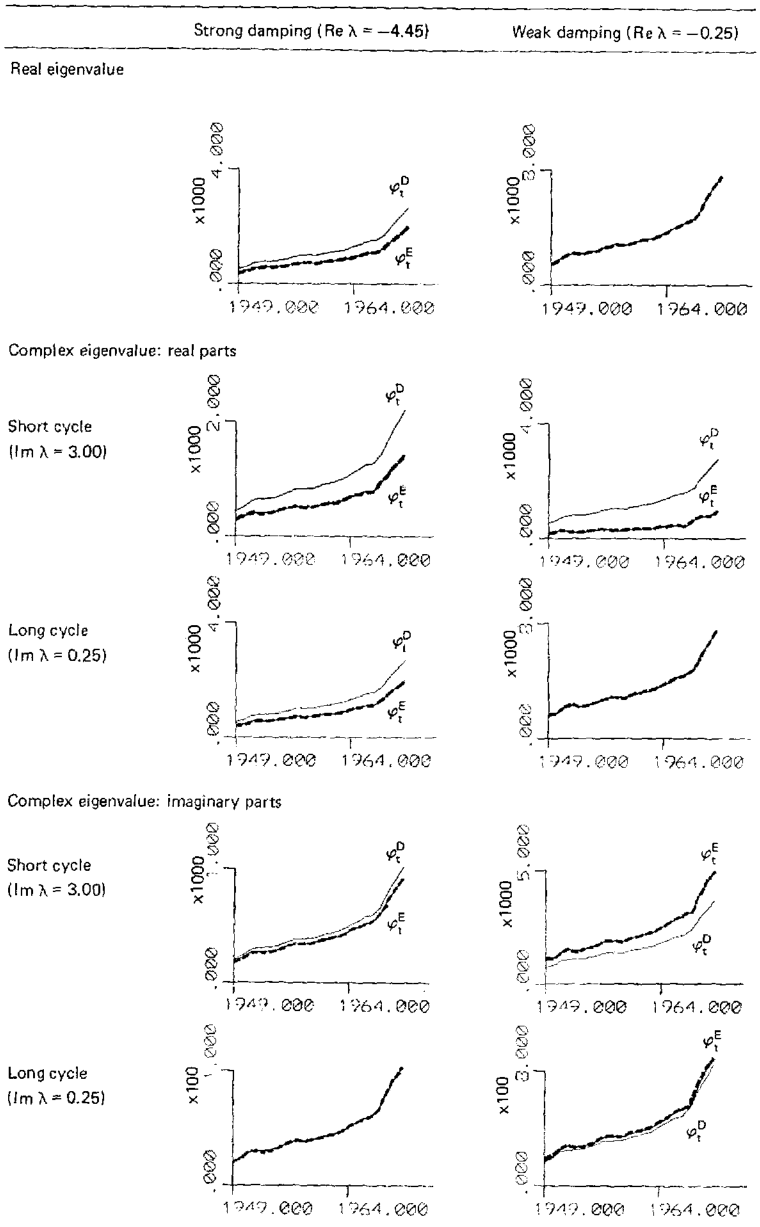


Chart 9. U.K. exports (visible trade). ----- denotes Φ_t.

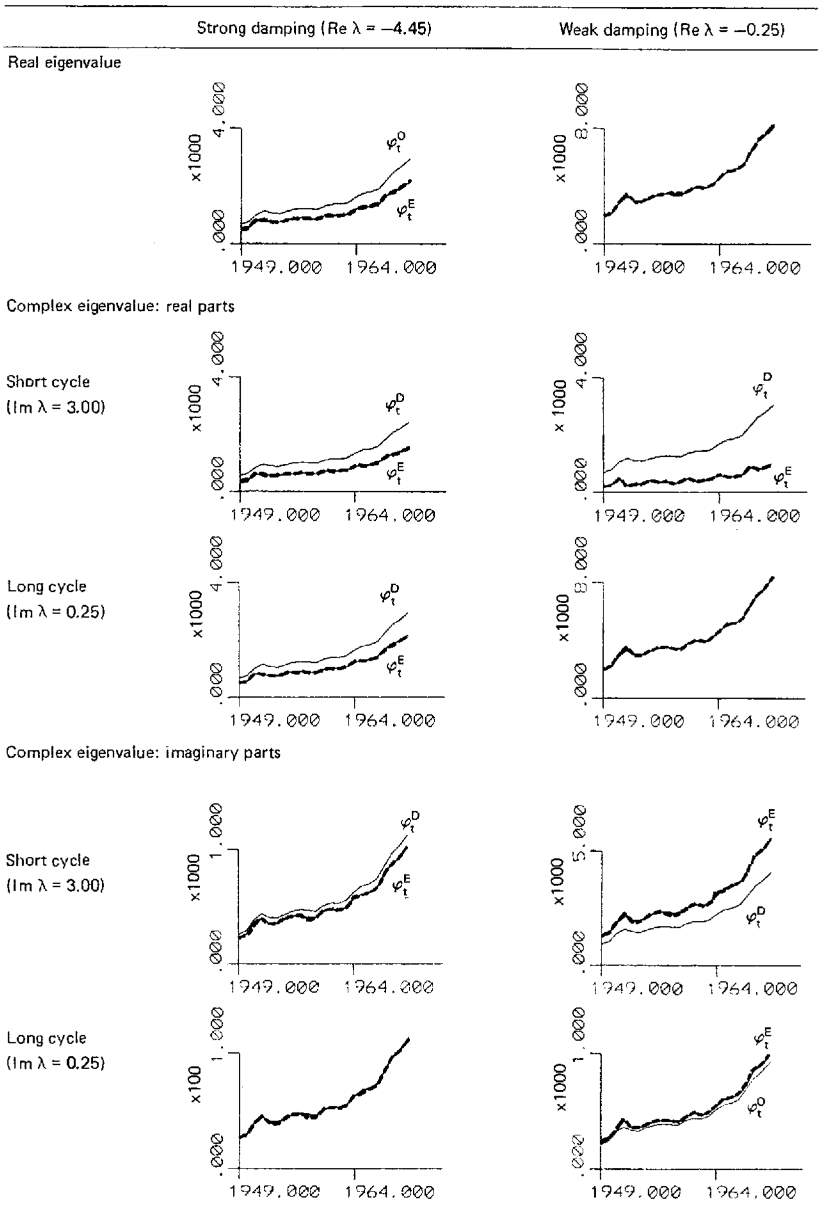


Chart 10. U.K. imports (visible trade). ——— denotes Φ_t.

a consistent bias for most eigenvalues, but Φ_t^E significantly underestimates Φ_t in an early year (1952), while in later years it appears to provide a satisfactory approximation. Turning to the data graphs given in the appendix we note that the Metals series (figure 8) displays definite irregularities. It is interesting that the irregularities sometimes disrupt Φ_t^E more than Φ_t^D as in the case of the early years. This must be caused by fact that Φ_t^E apportions different weights to the current and lagged observations whereas Φ_t^D does not. Thus, if the discrete (in the present case, annual) observations are not representative of the series in a particular interval and this happens to be accentuated by the weighting scheme in Φ_t^E, then the approximation Φ_t^E will be more severely disrupted than Φ_t^D because Φ_t^D depends on a simple average of the observations with equal weights.

Since the E-model approximation did not perform as well in the case of the metals series as for the other monthly series we decided to compute the smoothness coefficient (developed in section 3) for each series by running a regression of the form (16). The results are given below:

Series	Smoothness coefficient (R^2)
(iv) Index of Industrial Production	0.9890
(v) Registered Unemployed	0.7755
(vi) Import Price Index (Food)	0.9619
(vii) Import Price Index (Total)	0.9481
(viii) World Price Index (Metals)	0.8516
(ix) Exports	0.9947
(x) Imports	0.9862

Two series, Registered Unemployed and the Metals Index, have smoothness coefficients which are considerably smaller than those of the remaining series. If we accept that both these series display irregularities, then this would explain the performance of the E-model approximation in the case of Metals. But we are led to question why the E-model is apparently so much better than the D-model approximation for the Unemployment series. One answer that is meaningful in the context of our theory is that much of the apparent irregularity in the Unemployment series is caused by a seasonal component. While this seasonal component certainly affects, in this case, the size of the R^2 in the regression (16), it is less important as far as the performance of the approximations are concerned because the seasonal cycle is itself reasonably smooth (if not completely regular as in the case of the Industrial Production Index). The data graphs in the appendix seem to support this hypothesis.

5. *Conclusions and some further computations*

The results of sections 3 and 4 indicate that for most of the economic series we have considered in this paper the E-model gives a more reliable approximation than the D-model. When the system eigenvalue has large real and imaginary parts, the relatively poor performance of the D-model approximation is particularly evident and this confirms the theory of section 2. One of our conclusions, therefore, is that the theory we have developed on the assumption that the exogenous series are reasonably smooth does seem to provide a useful guide to the relative performance of the approximations in practical work with observed series.

For two economic series, we found that the E-model approximation did not perform as well as our theory might suggest. Using an indicator of the smoothness of a series, we found that these particular series appeared to more irregular on the whole than the others. This result is in agreement with the asymptotic theory developed in [2] where it was established that the order of magnitude of the asymptotic bias of estimators derived from the E-model depends on the smoothness properties of the exogenous series.

To make the results of this paper more useful to empirical researchers in this area, we must make certain recommendations. Our first recommendation is that, in general, the E-model is worth estimating because it is likely to be more reliable than the D-model, particularly when there is a trade cycle mechanism in our model involving moderate to strong damping factors.

Our second recommendation is that a researcher who is doubtful whether his exogenous series are very smooth should compute the smoothness coefficient developed in section 3 and assess whether its value implies that the E-model may not perform satisfactorily. To help in this assessment, we have carried out some further computations.

From the statistics given in tables 1–10 we calculated, for each group of eigenvalues, the ratio

$$\sum_{j=1}^{30}\left\{\sum_{t=3}^{T}|\Phi_{tj}-\Phi_{tj}^{E}|^2\right\}^{\frac{1}{2}}\Bigg/\sum_{j=1}^{30}\left\{\sum_{t=3}^{T}|\Phi_{tj}-\Phi_{tj}^{D}|^2\right\}^{\frac{1}{2}}, \tag{18}$$

where the subscript j refers to the eigenvalue λ_j in a particular group. In (18) the vertical bars indicate that we are taking the moduli of the deviations $\Phi_{tj} - \Phi_{tj}^E$ and $\Phi_{tj} - \Phi_{tj}^D$, so that in computing (18) the real and imaginary parts of these deviations are considered together. For each table, 12 such ratios were calculated corresponding to the different eigenvalue groups, and we used the geometric mean of these 12 ratios as an indicator of the relative performance of the E-model and D-model approximations for this series. We denote this geometric mean by G_l where l ($l = 1, \ldots, 10$) refers to the series.

To investigate the relationship between the relative performance of the approximations and the smoothness of the series that is suggested by our theory, we decided to carry out a simple linear regression of $\{G_l: l = 1, \ldots, 10\}$ on the smoothness coefficients which we denote by H_l ($l = 1, \ldots, 10$). This regression resulted in the relationship

$$G_l = \underset{(0.4531)}{2.3690} - \underset{(0.5018)}{2.3818}\, H_l, \quad R^2 = 0.7379, \tag{19}$$

where we give the estimated standard errors in brackets.

We notice that the coefficient of H_l in (19) is significantly less than zero which is consistent with the hypothesis our theory favours, and we have explained 74% of the variation in G_l by this regression. We can use (19) to compute a critical value of H_l, below which we cannot recommend the use of the E-model for estimation purposes. This critical value is

$$H_l = 0.5748, \tag{20}$$

for, when the smoothness coefficient H_l is less than this value, then according to (19) we can expect the ratio (18) to be greater than unity. Thus, if an empirical investigator finds that his exogenous series have a smoothness coefficient which is less than (20) then our results indicate that, in this case, it may not be worth while estimating the E-model.

We can put this recommendation to a simple test. As we remarked in section 1, a finite sample experiment[9] has already been carried out with artificial data and the E-model estimates in this experiment turned

[9] See [3].

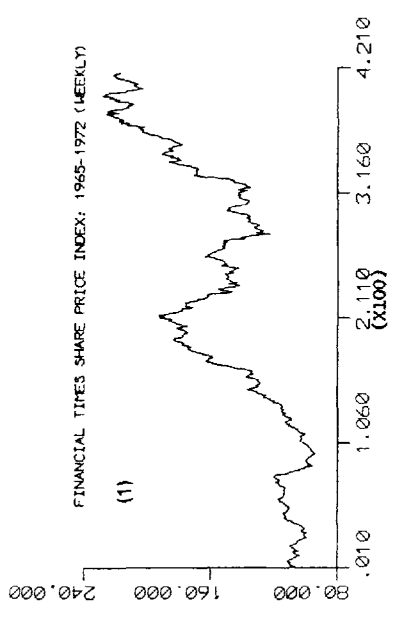
out to be somewhat disappointing. To test whether this outcome might have been forecast by the above results, we computed the smoothness coefficients for the two exogenous series used in this sampling experiment. We found the following:

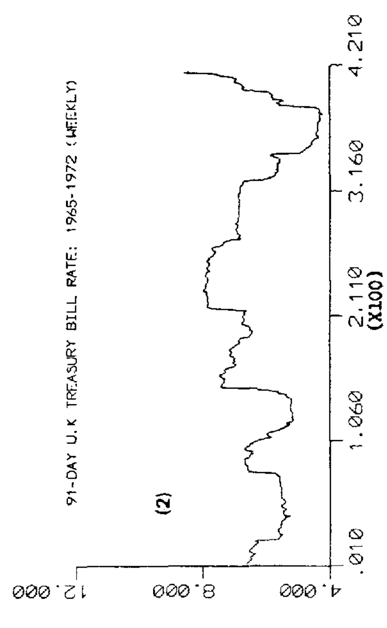
Series	Smoothness coefficient (R^2)
Imports (Artificial Data)	0.2351
Exports (Artificial Data)	0.4187

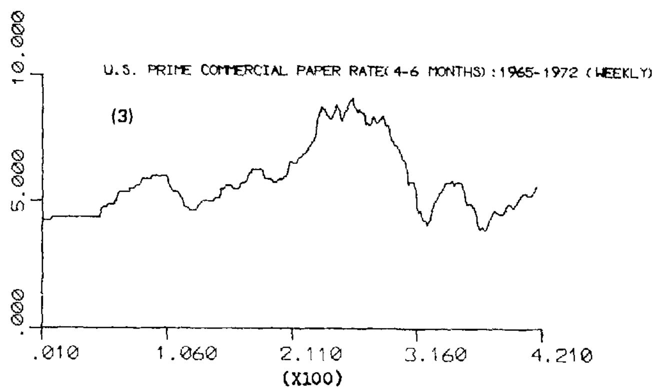
The smoothness coefficients for both series are well below the lower bound (20) we have prescribed. Thus, our recommendation that the E-model may not be worth estimating when the smoothness coefficient is below (20) appears to be justified in this case.

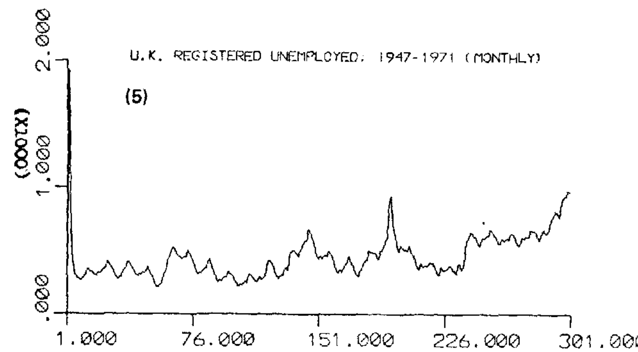
Appendix

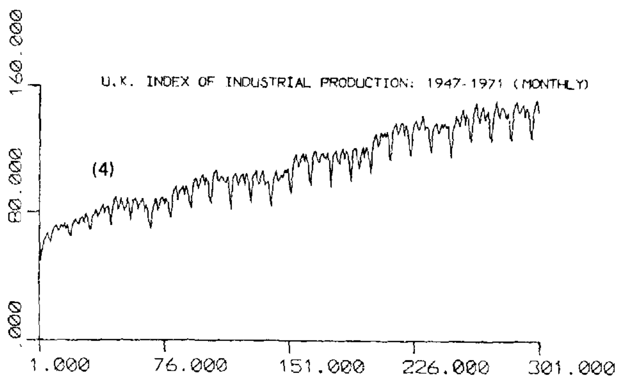
Figures 1–10. Units measured on the x-axis are weeks in the case of weekly data, months in the case of monthly data.

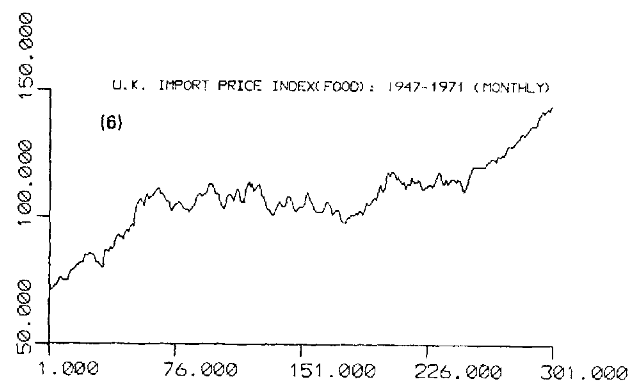


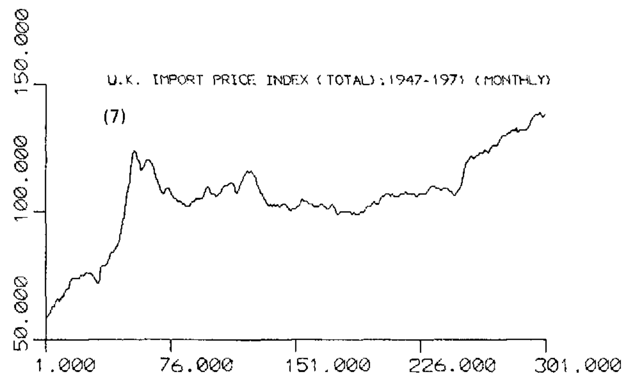


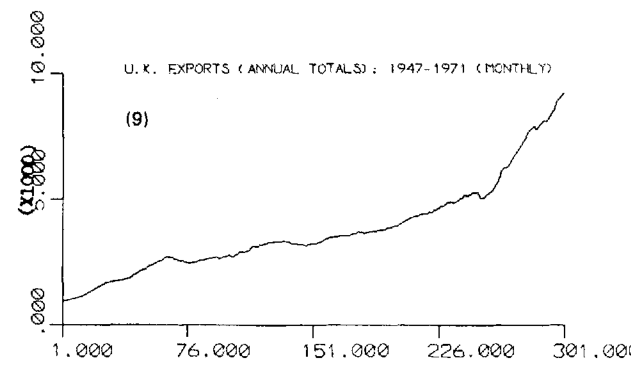

U.S. PRIME COMMERCIAL PAPER RATE(4-6 MONTHS) : 1965-1972 (WEEKLY)
(3)
10.000
5.000
.000
.010
1.060
2.110
3.160
4.210
(X100)

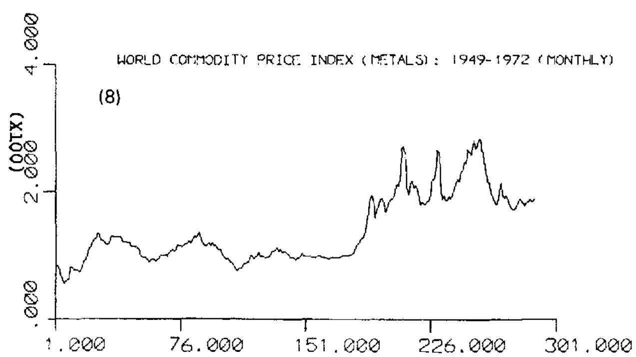

U.K. REGISTERED UNEMPLOYED: 1947-1971 (MONTHLY)
(5)
2.000
1.000
.000
(X1000)
1.000
76.000
151.000
226.000
301.000

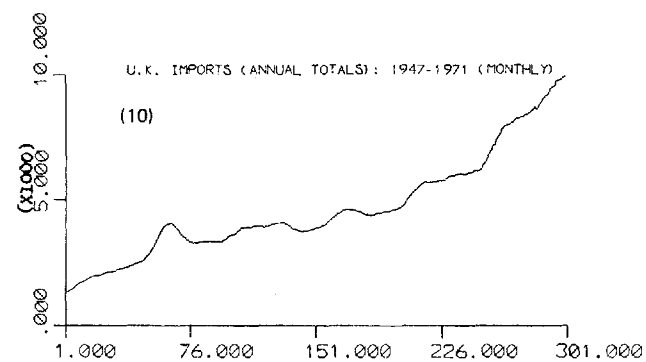

U.K. INDEX OF INDUSTRIAL PRODUCTION: 1947-1971 (MONTHLY)
(4)
160.000
80.000
.000
1.000
76.000
151.000
226.000
301.000


U.K. IMPORT PRICE INDEX(FOOD) : 1947-1971 (MONTHLY)
(6)
150.000
100.000
50.000
1.000
76.000
151.000
226.000
301.000


U.K. IMPORT PRICE INDEX (TOTAL); 1947-1971 (MONTHLY)
(7)
150.000
100.000
50.000
1.000
76.000
151.000
226.000
301.000


U.K. EXPORTS (ANNUAL TOTALS): 1947-1971 (MONTHLY)
(9)
10.000
5.000
(X1000)
.000
1.000
76.000
151.000
226.000
301.000


WORLD COMMODITY PRICE INDEX (METALS): 1949-1972 (MONTHLY)
(8)
4.000
2.000
(X100)
.000
1.000
76.000
151.000
226.000
301.000


U.K. IMPORTS (ANNUAL TOTALS): 1947-1971 (MONTHLY)
(10)
10.000
5.000
(X1000)
.000
1.000
76.000
151.000
226.000
301.000

References

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