Incidental Trends and the Power of Panel Unit Root Tests

Hyungsik Roger Moon
Department of Economics
University of Southern California

Benoit Perron
Département de sciences économiques and CRDE
Université de Montréal

Peter C.B. Phillips
Cowles Foundation, Yale University
University of Auckland & University of York

June 2003

Abstract

The asymptotic local power of various panel unit root tests is investigated. The asymptotic power envelope is obtained under homogeneous and heterogeneous alternatives and compared with asymptotic power functions of the pooled t- test, the Ploberger-Phillips (2002) test and a point optimal test. All three tests have significant asymptotic power in neighborhoods of unity that are of order $n^{-1/4}T^{-1}$ and $n^{-1/2}T^{-1}$, depending on whether or not incidental trends are extracted from the panel data. Some simulations examining finite sample performance of the tests are reported.

1 Introduction

In the past decade, much research has been conducted on panels in which both the cross-sectional and time dimensions are large. Testing for a unit root in such panels has been a major focus of this research. For example, Quah (1994), Levin *et al* (2002), Im *et al* (1997), Maddala and Wu (1999), and Choi (2001) have all proposed various tests. These studies derived the limit theory for the tests only under the null hypothesis of a common panel unit root and power properties were investigated by simulation.

The asymptotic local power properties of some panel unit root tests have become known quite recently. Moon and Perron (2003a) show that *without incidental trends* in the panel, their panel unit root test which is based on a t-ratio type statistic has significant asymptotic local power in a neighborhood of unity that shrinks to the null at the rate of $n^{-1/2}T^{-1}$ (where n and T denote the size of the cross-section and time dimensions, respectively). However, *in the presence of incidental trends*, the t-ratio type test statistic constructed from ordinary least squares (OLS) detrended data does not have any power in a $n^{-\kappa}T^{-1}$- neighborhood of unity with $\kappa > 1/6$. For a panel with incidental trends, Ploberger and Phillips (2002) proposed an optimal invariant panel unit root test that maximizes average local power. They show that the optimal invariant test has asymptotic local power in a neighborhood of unity that shrinks at the rate $n^{-1/4}T^{-1}$.

The present study makes three contributions. First, the local asymptotic power envelope of the panel unit root testing problem is derived for three scenarios: (i) with no fixed effects; (ii) with fixed effects that are parameterized by heterogeneous intercept terms (deemed incidental parameters); and (iii) with fixed effects that are parameterized by heterogeneous linear deterministic trends (deemed incidental trends). For cases (ii) and (iii) we restrict the class of tests to be invariant with respect to the incidental parameters and trends. We show that in cases (i) and (ii), the power envelope is defined within $n^{-1/2}T^{-1}$- neighborhoods of unity and that it depends on the first two moments of the local to unity parameters. On the other hand, in case (iii), the power envelope is defined within $n^{-1/4}T^{-1}$- neighborhoods of unity and it depends on the first four moments of the local to unity parameters.

Second, we derive the asymptotic local power of some existing panel unit root tests and compare these to the power envelope. For case (i), we investigate the t-ratio statistics studied by Quah (1994), Levin *et al*, and Moon and Perron (2003a). For case (ii), we investigate a modified t-ratio statistic that is asymptotically equivalent to the test proposed by Levin *et al*. For case (iii), we compare the optimal invariant test proposed by Ploberger and Phillips (2002) and the LM test proposed by Moon and Phillips (2002). First, we show that in all three cases the existing tests do not achieve the optimal power. Next, when the alternative hypothesis is homogeneous across individuals, it is shown that some tests (the $t - test$ in case (i) and the optimal invariant test by Ploberger and Phillips (2002) in cases (ii) and (iii)) do achieve the power envelope and are uniformly most powerful.

Third, we propose a simple point optimal invariant panel unit root test for each case. These tests are optimal when the alternative hypothesis is homogeneous.

The paper is organized as follows. Section 2 lays out the model, the hypotheses to test, and the assumptions maintained throughout the paper. Section 3 studies the model where there are no fixed effects (or fixed effects are known), develops the power envelope, gives a point optimal test and performs some power comparisons. Sections 4 and 5 perform similar analyses for panel models with fixed effects and incidental trends. Section 6 reports some simulations comparing the finite sample properties of the main tests studied in Section 5. Section 7 concludes and the Appendix contains technical derivations and proofs.

2 Model

The observed panel z_{it} is assumed to be generated by the following component model

$$\begin{aligned} z_{it} &= b_i' g_t + y_{it} \\ y_{it} &= \rho_i y_{it-1} + u_{it},\ i = 1, ..., n;\ t = 1, ..., T, \end{aligned} \tag{1}$$

where u_{it} is a mean zero error, $g_t = (1, t)'$, and $b_i = (b_{i0}, b_{i1})'$.

The focus of interest is the problem of testing for the presence of a common unit root in the panel against local alternatives when both n and T are large. For a local alternative specification we assume that

$$\rho_i = 1 - \frac{\theta_i}{n^\kappa T} \text{ for some constant } \kappa > 0, \tag{2}$$

where θ_i is a sequence of iid random variables. The main goal of the paper is to find efficient tests for the null hypothesis

$$\mathbb{H}_0 : \ \theta_i = 0 \text{ a.s. } (i.e., \rho_i = 1) \text{ for all } i, \tag{3}$$

against the alternative

$$\mathbb{H}_1 : \theta_i \neq 0 \ \ (i.e., \ \rho_i \neq 1) \ \text{ for some } i's. \tag{4}$$

A common special case of interest for the alternative hypothesis $\mathbb{H}_1$ is

$$\mathbb{H}_2 : \theta_i = \theta > 0 \text{ for all } i, \tag{5}$$

where the local to unity coefficients take on a common value $\theta > 0$ for all i.In this case, the series are then locally stationary, that is $\rho_i = \rho = 1 - \frac{\theta}{n^\kappa T} < 1$ for all i.

In (1) the nonstationary panel z_{it} has two different types of trends. The first component $b_i' g_t$ is a deterministic linear trend that is heterogeneous across individuals i. This component characterizes individual effects in the panel. The second component y_{it} is a stochastic trend or near unit-root process with ρ_i close to unity.

The following sections look at three different cases. In the first case b_{i0} and b_{1i} are observable, so that y_{it} is observable. This is essentially a situation where there are no fixed effects in the panel. The second case arises when b_{0i} are unobserved but b_{1i} are observable. In this case, the panel data z_{it} contain fixed effects that are parameterized by heterogeneous intercept terms b_{0i}, which are incidental parameters to be estimated. The third case arises when both b_{0i} and b_{1i} are unobserved, so the panel contains fixed effects that are parameterized by heterogeneous linear deterministic trends, $b_{0i} + b_{1i} t$ where both sets of parameters b_{0i} and b_{1i} are to be estimated.

Before proceeding, we introduce the following notation. Define

$$\begin{aligned} z_t &= (z_{1t}, ..., z_{nt})',\ y_t = (y_{1t}, ..., y_{nt})',\ u_t = (u_{1t}, ..., u_{nt})' \\ Z &= (z_1, ..., z_T),\ Y = (y_1, ..., y_T),\ Y_{-1} = (y_0, y_1, ..., y_{T-1}),\ U = (u_1, ..., u_T), \end{aligned}$$

so the $(i,t)^{th}$ elements of Z, Y, Y_{-1}, and U are z_{it}, y_{it}, y_{it-1}, and u_{it}, respectively. Define

$$\begin{aligned} G_0 &= (1, ..., 1)' : \ T- \text{ vector of ones, } G_1 = (1, 2, ..., T)', \\ G &= (G_0, G_1) = (g_1, ..., g_T)', \end{aligned}$$

and

$$\begin{aligned} \beta_0 &= (b_{01}, ..., b_{0n})', \ \beta_1 = (b_{11}, ..., b_{1n})', \\ \beta &= (\beta_0, \beta_1) = (b_1, ..., b_n)', \end{aligned}$$

Let $\underline{Z}_i$, $\underline{Y}_i$, $\underline{Y}_{-1,i}$, and $\underline{U}_i$ denote the transpose of the i^{th} row of Z, Y, Y_{-1}, and U, respectively, and write the model in matrix forma as

$$\begin{aligned} Z &= \beta G' + Y, \\ Y &= \rho Y_{-1} + U, \end{aligned}$$

where $\rho = diag(\rho_1, ..., \rho_n)$.

Assumption 1 $u_{it} \sim iid\left(0, \sigma^2\right)$ *with finite fourth moment for* $i = 1, 2...., n$ *and over* $t = 1, 2, ..., T$.

Assumption 2 *The initial observations* y_{i0} *are iid with* $E\left|y_{i0}\right|^{\nu} < \infty$ *for some* $\nu > 2$ *and are independent of* u_{it}, $t \geq 1$ *for all* i.

Assumption 3 $\frac{1}{T} + \frac{1}{n} + \frac{n^{3/4}}{T} \to 0$.

Assumption 1 imposes a restrictive error structure that will often be unrealistic. The main reason for using it here is to facilitate analytical derivations and focus on more essential elements in power calculations. In Section 4 we briefly discuss how it may be relaxed.

The error variance σ^2 is usually unknown. Most tests depend on sutable estimates of σ^2 and in what follows we may replace σ^2 with any estimator $\hat{\sigma}^2$ that is consistent under both the null and alternative hypotheses. An example of such an estimator is provided in Moon and Phillips (2002). They show that $\hat{\sigma}^2 - \sigma^2 = O_p\left(\frac{1}{\sqrt{T}} \max\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right)\right)$, where $\hat{\sigma}^2 = \frac{1}{nT} tr\left(\hat{e}'\hat{e}\right)$ and $\hat{e}$ is the matrix of residuals from a pooled autoregression on demeaned or detrended data[1]. For our purpose in this paper, it is convenient to make the following generic assumption about the variance estimate $\hat{\sigma}^2$..

Assumption 4 $\hat{\sigma}^2 - \sigma^2 = O_p\left(\frac{1}{\sqrt{T}} \max\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right)\right)$.

[1] See Lemma 2 of Moon and Phillips (2002).

3 Without Fixed Effects

This section investigates the model in which $b_i' g_t$ is observable or equivalently that $g_t = 0$ and y_{it} is observable. We consider local neighborhoods of unity that shrink at the rate of $\frac{1}{n^{1/2}T}$ and one sided alternatives, as indicated in the following assumptions.

Assumption 5 $\kappa = 1/2$ *in (2).*

Assumption 6 θ_i *is a sequence of iid random variables on a non-negative bounded support* $[0, M_\theta]$ *for some* $M_\theta \geq 0$.

Let $\mu_{\theta,k} = E\left(\theta_i^k\right)$. The assumption of a bounded support for θ_i is made for convenience, and could be relaxed at the cost of stronger moment conditions. It is also convenient to assume that the θ_i are identically distributed, and this assumption could be relaxed as long as cross sectional averages of the moments $\frac{1}{n}\sum_{i=1}^{n} E\left(\theta_i^k\right)$ have limits like $\mu_{\theta,k}$.

According to Assumption 6, $\theta_i \geq 0$ for all i, so that $\rho_i \leq 1$. In this case, the null hypothesis of a unit root in (3) is equivalent to $\mu_{\theta,1} = 0$ or $M_\theta = 0$ (*i.e.* $\theta_i = 0$ *a.s.*), and the alternative hypothesis in (7) implies $\mu_{\theta,1} > 0$. Hence, in this section we set the hypotheses in terms of the first moment θ_i as follows:

$$\mathbb{H}_0 : \mu_{\theta,1} = 0, \tag{6}$$

and

$$\mathbb{H}_1 : \mu_{\theta,1} > 0. \tag{7}$$

To test these hypotheses, Moon and Perron (2003a) propose t - ratio tests based on a modified pooled OLS estimator of the autoregressive coefficient and show that they have significant asymptotic local power in neighborhoods of unity shrinking at the rate $\frac{1}{\sqrt{n}T}$. This section proposes a uniformly most powerful test for $\mathbb{H}_0$. We first derive the (asymptotic) power envelope and show that the power function of a point optimal test for $\mathbb{H}_0$ achieves the envelope for the hypotheses above. We then derive compare the asymptotic local power of this point-optimal test with that of the Moon and Perron test.

3.1 Power Envelope

The power envelope is found by computing the upper bound of the power of all point optimal tests for each local alternative. To proceed, we define

$$\rho_{c_i} = 1 - \frac{c_i}{n^{1/2}T},$$

where c_i is an *iid* sequence of random variables on $[0, M_c]$ for some $M_c > 0$. Denote by $\mu_{c,k}$ the k^{th} raw moment of c_i, *i.e.*, $\mu_{c,k} = E\left(c_i^k\right)$. Let

$$\mathbb{C} = diag\left(c_1, ..., c_n\right) \tag{8}$$

and

$$\Delta_{\mathbb{C}} = diag\left(1 - \rho_{c_i} L\right), \tag{9}$$

where L denote the lag operator. Define

$$\Delta_{\mathbb{C}} Y = \left(y_0, \Delta_{\mathbb{C}} y_1 ..., \Delta_{\mathbb{C}} y_t, ..., \Delta_{\mathbb{C}} y_T\right).$$

so that for $t \geq 1$, the $(i,t)^{th}$ element of $\Delta_{\mathbb{C}} Y$ is $y_{it} - y_{it-1} + \frac{c_i}{n^{1/2}T} y_{it-1}$, a quasi difference of y_{it}. For notational simplicity, let $\Delta = \Delta_0$.

Define

$$V_{nT}(\mathbb{C}) = \frac{1}{\hat{\sigma}^2}\left[tr\left(\Delta_{\mathbb{C}} Y \left(\Delta_{\mathbb{C}} Y\right)'\right) - tr(\Delta Y \left(\Delta Y\right)')\right] - \frac{1}{2}\mu_{c,2}.$$

The statistic $V_{nT}(\mathbb{C})$ is the likelihood ratio statistic of the null hypothesis $\rho_i = 1$ against an alternative hypothesis $\rho_i = \rho_{c_i}$ for $i = 1, ..., n$. According to the Neyman-Pearson lemma, rejecting the null hypothesis for small values of $V_{nT}(\mathbb{C})$ is the most powerful test of the null hypothesis $\mathbb{H}_0$ against the alternative hypothesis $\rho_i = \rho_{c_i}$. When the alternative hypothesis is given by $\mathbb{H}_1$ $(c_i = \theta_i)$ the test is a point optimal test (see, e.g., King (1988)). Let $\Psi_{nT}(\mathbb{C})$ be the test that rejects $\mathbb{H}_0$ for small values of $V_{nT}(\mathbb{C})$.

Since $\Delta y_{it} = -\frac{\theta_i}{n^{1/2}T} y_{it-1} + u_{it}$ under Assumption 5,

$$\begin{aligned}
& V_{nT}(\mathbb{C}) \\
= & \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n}\left[y_{i0}^2 + \sum_{t=1}^{T}\left(\Delta_{c_i} y_{it}\right)^2\right] - \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n}\left[y_{i0}^2 + \sum_{t=1}^{T}\left(\Delta y_{it}\right)^2\right] - \frac{1}{2}\mu_{c,2} \\
= & \frac{2}{n^{1/2}T\hat{\sigma}^2}\sum_{i=1}^{n} c_i \sum_{t=1}^{T}\Delta y_{it} y_{it-1} + \frac{1}{nT^2\hat{\sigma}^2}\sum_{i=1}^{n} c_i^2 \sum_{t=1}^{T} y_{it-1}^2 - \frac{1}{2}\mu_{c,2} \\
= & -\frac{2}{nT^2\hat{\sigma}^2}\sum_{i=1}^{n} c_i\theta_i \sum_{t=1}^{T} y_{it-1}^2 + \frac{2}{n^{1/2}T\hat{\sigma}^2}\sum_{i=1}^{n} c_i^2 \sum_{t=1}^{T} u_{it} y_{it-1} \\
& + \frac{1}{nT^2\hat{\sigma}^2}\sum_{i=1}^{n} c_i^2 \sum_{t=1}^{T} y_{it-1}^2 - \frac{1}{2}\mu_{c,2}.
\end{aligned}$$

Direct calculation shows that under Assumptions 1 – 4,

$$\begin{aligned}
-\frac{2}{nT^2\hat{\sigma}^2}\sum_{i=1}^{n}\sum_{t=1}^{T} c_i\theta_i y_{it-1}^2 & \to_p \quad -E\left(c_i\theta_i\right), \\
\frac{1}{nT^2\hat{\sigma}^2}\sum_{i=1}^{n} c_i^2 \sum_{t=1}^{T} y_{it-1}^2 & \to_p \quad \frac{1}{2}\mu_{c,2},
\end{aligned}$$

and

$$\frac{2}{n^{1/2}T\hat{\sigma}^2}\sum_{i=1}^{n} c_i \sum_{t=1}^{T} u_{it} y_{it-1} \Rightarrow N\left(0, 2\mu_{c,2}\right),$$

thereby giving the following result.

Theorem 7 *Suppose that Assumptions 1 – 6 hold. Then,*

$$V_{nT}(\mathbb{C}) \Rightarrow N\left(-E\left(c_i\theta_i\right), 2\mu_{c,2}\right).$$

The asymptotic critical values of the test $\Psi_{nT}(\mathbb{C})$ can be readily computed. Let $\bar{z}_\alpha$ denote the $(1-\alpha)-$ quantile of the standard normal distribution, *i.e.*, $P(Z \leq -\bar{z}_\alpha) = \alpha$, where $Z \sim N(0,1)$. Then, the size α asymptotic critical value $\psi(\mathbb{C}, \alpha)$ of the test $\Psi_{nT}(\mathbb{C})$ is

$$\psi(\mathbb{C}, \alpha) = -\sqrt{2\mu_{c,2}}\bar{z}_\alpha,$$

and its asymptotic local power is

$$\Phi\left(\frac{E\left(c_i\theta_i\right)}{\sqrt{2\mu_{c,2}}} - \bar{z}_\alpha\right), \tag{10}$$

where $\Phi(x)$ is the cumulative distribution function of Z.

From (10), it is easy to find the power envelope, i.e., the values of c_i for which power is maximized. By the Cauchy-Schwarz inequality

$$\Phi\left(\frac{E\left(c_i\theta_i\right)}{\sqrt{2\mu_{c,2}}} - \bar{z}_\alpha\right) \leq \Phi\left(\sqrt{\frac{\mu_{\theta,2}}{2}} - \bar{z}_\alpha\right),$$

and the upper bound of the power $\Phi\left(\sqrt{\frac{\mu_{\theta,2}}{2}} - \bar{z}_\alpha\right)$ is achieved with $c_i = \theta_i$. Then, by the Neyman-Pearson lemma, $\Phi\left(\sqrt{\frac{\mu_{\theta,2}}{2}} - \bar{z}_\alpha\right)$ is the power envelope. We have the following theorem.

Theorem 8 *Assume that the trends $b_i'g_t$ in* (1) *are known. Suppose that Assumptions 1 – 6 hold. Then, the power envelope for testing for $\mathbb{H}_0$ in* (3) *against $\mathbb{H}_1$ in* (4) *is $\Phi\left(\sqrt{\frac{\mu_{\theta,2}}{2}} - \bar{z}_\alpha\right)$, where $\mu_{\theta,2} = E\left(\theta_i^2\right)$ and $\bar{z}_\alpha$ is the $(1-\alpha)-$ quantile of the standard normal distribution.*

Note that a necessary condition for attaining the power envelope is $c_i = \theta_i$ *a.s.*, which in turn requires that the support of c_i be the same as the support of θ_i, i.e., $M_c = M_\theta$.

3.2 Power Comparison

3.2.1 The $t-$ ratio Test

We start by investigating the $t-$ ratio test of Quah (1994), Levin *et al* (2002), and Moon and Perron (2003a), which is based on the pooled OLS estimator[2].

[2] When the error term u_{it} is serially correlated, one can use a modified version of the pooled OLS estimator. For details of this modification, refer to Moon and Perron (2003a).

Let

$$\hat{\rho} = \frac{\sum_{i=1}^{n}\sum_{t=1}^{T} y_{it}y_{it-1}}{\sum_{i=1}^{n}\sum_{t=1}^{T} y_{it-1}^2},$$

be the pooled OLS estimator and the corresponding t statistic

$$t = \frac{\hat{\rho}-1}{\hat{\sigma}\sqrt{\sum_{i=1}^{n}\sum_{t=1}^{T} y_{it-1}^2}}.$$

Under the conditions assumed above, we have

$$t \Rightarrow N\left(-\frac{\mu_{\theta,1}}{\sqrt{2}}, 1\right).$$

The power of the t test with size α is then

$$\Phi\left(\frac{\mu_{\theta,1}}{\sqrt{2}} - \bar{z}_\alpha\right). \tag{11}$$

Remarks

(a) By the Cauchy-Schwarz inequality, it is straightforward to show that

$$\Phi\left(\frac{\mu_{\theta,1}}{\sqrt{2}} - \bar{z}_\alpha\right) \leq \Phi\left(\sqrt{\frac{\mu_{\theta,2}}{2}} - \bar{z}_\alpha\right). \tag{12}$$

In view of (12), the t ratio test achieves optimal power only when the alternative is homogeneous as in $\mathbb{H}_2$, that is when $\theta_i = \theta$ *a.s.*, so that $E(\theta_i) = \sqrt{E(\theta_i^2)}$. Otherwise, the power of the t ratio test is strictly less than the optimal power. This implies that $t-$ ratio test is uniformly most powerful test for testing $\mathbb{H}_0$ against $\mathbb{H}_2$ but not against $\mathbb{H}_1$. The result is not surprising since the t ratio test is constructed based on the pooled OLS estimator and pooling is efficient under the homogeneous alternative.

(b) Notice from (10) that the asymptotic local power envelope is determined by $\mu_{\theta,1}$, the mean of the local to unity parameters θ_i. In the given formulation, the local alternative is restricted to be one sided in Assumption 6. Allowing for two-sided alternatives opens the possibility that $\mu_{\theta,1} = 0$ even under the alternative hypothesis, in which case the power of the pooled $t-$ test is equivalent to size.

3.2.2 A Common-Point Optimal Test with $c_i = c$

As shown earlier, to achieve the power envelope, one needs to choose $c_i = \theta_i$ a.s. for $\Psi_{nT}(\mathbb{C})$. Denote this test $\Psi_{nT}(\not\leq)$. Of course, this test $\Psi_{nT}(\not\leq)$ is infeasible because it is not possible to identify the distribution of θ_i in the panel and generate a sequence from its distribution; and, if the θ_i were known, there would of course be no need to test the null of a panel unit root.

One way of implementing the test $\Psi_{nT}(\mathbb{C})$ is to use randomly generated $c_i's$ from some domain that is considered relevant. The variates c_i are independent of θ_i and the power of the test $\Psi_{nT}(\mathbb{C})$ is

$$\Phi\left(\frac{\mu_{c,1}\mu_{\theta,1}}{\sqrt{2\mu_{c,2}}} - \bar{z}_\alpha\right). \tag{13}$$

Since $\mu_{c,1} \leq \sqrt{\mu_{c,2}}$, the power (13) is bounded by

$$\Phi\left(\frac{\mu_{\theta,1}}{\sqrt{2}} - \bar{z}_\alpha\right), \tag{14}$$

which is achieved if we choose $c_i = c$, where c is any positive constant. We denote this test $\Psi_{nT}(c)$.

Remarks

(a) Not surprisingly, the power (14) of the test $\Psi_{nT}(c)$ is identical to that of the t - ratio test in the previous section. Of course, both tests are based on the homogeneous alternative hypothesis.

(b) Note that the power of the test $\Psi_{nT}(c)$ does not depend on c. The test is optimal against the special homogeneous alternative hypothesis $\mathbb{H}_2$ for any choice of c. This result is in contrast to the power of the point optimal test for unit root time series in Elliot *et al* (1996), where the power of the test does depend on the value of c. The reason is that the local alternative in the panel unit root case is $\rho_{c_i} = 1 - \frac{c}{n^{1/2}T}$ which is closer to the null hypothesis than the alternative $\rho_{c_i} = 1 - \frac{c}{T}$ that applies in the case where there is only time series data. In effect, for the panel point optimal test under homogeneity, it suffices to use any common local alternative in setting up the test.

4 Fixed Effects I: Incidental Parameters Case

The model we consider in this section assumes that the fixed effects $b_i g_t = b_{i0}$, so that $g_t = 1$ or that the incidental trend term $b_{1i}t$ is known but the incidental parameter term b_{i0} is unknown. In this case, the model has the matrix form

$$Z = \beta_0 G_0' + Y.$$

4.1 Power Envelope

This section derives the power envelope of panel unit root tests for $\mathbb{H}_0$ that are invariant to the transformation $Z \to Z + \beta_0^* G_0'$ for arbitrary β_0^*. Recall the definition of the notation $\Delta_{\mathbb{C}}$ in (9). Define $\Delta_{\mathbb{C}} Z = (z_0, \Delta_{\mathbb{C}} z_1, ..., \Delta_{\mathbb{C}} z_T)$ and $\Delta_{\mathbb{C}} \beta_0 G_0' = (\beta_0, \Delta_{\mathbb{C}} \beta_0, ..., \Delta_{\mathbb{C}} \beta_0)$. Let

$$L_{nT}(\mathbb{C}, \beta_0) = tr(\Delta_{\mathbb{C}} Z - \Delta_{\mathbb{C}} \beta_0 G_0')(\Delta_{\mathbb{C}} Z - \Delta_{\mathbb{C}} \beta_0 G_0')'.$$

A (Gaussian) point optimal invariant test statistic for this fixed effects I case can be constructed as follows (see, for example, Lehmann (1959), Dufour and King (1991), and Elliott *et al* (1996)):

$$V_{fe1,nT}(\mathbb{C}) = \frac{1}{\hat{\sigma}^2} \left[\min_{\beta_0} L_{nT}(\mathbb{C}, \beta_0) - \min_{\beta} L_{nT}(0, \beta_0) \right] - \frac{1}{2} \mu_{c,2}$$

For given $c_i's$, the point optimal invariant test, say $\Psi_{fe1,nT}(\mathbb{C})$, rejects the null hypothesis for small values of $V_{fe1,nT}(\mathbb{C})$.

Letting $\hat{b}_{0i}(c_i) = (\Delta_{c_i} G_0' \Delta_{c_i} G_0)^{-1} (\Delta_{c_i} G_0' \Delta_{c_i} \underline{Z}_i)$ and $\hat{\underline{Y}}_i(c_i) = \underline{Z}_i - G_0 \hat{b}_{0i}(c_i) = \underline{Y}_i - G_0 \left(\hat{b}_{0i}(c_i) - b_{0i} \right)$, we can rewrite $V_{fe1,nT}(\mathbb{C})$ as

$$\begin{aligned}
& V_{fe1,nT}(\mathbb{C}) \\
= & \frac{1}{\hat{\sigma}^2} \sum_{i=1}^{n} \left[\begin{array}{c} \left(\hat{\underline{Y}}_i(c_i) - \rho_{c_i} \hat{\underline{Y}}_{-1,i}(c_i) \right)' \left(\hat{\underline{Y}}_i(c_i) - \rho_{c_i} \hat{\underline{Y}}_{-1,i}(c_i) \right) \\ - \left(\hat{\underline{Y}}_i(0) - \hat{\underline{Y}}_{-1,i}(0) \right)' \left(\hat{\underline{Y}}_i(0) - \hat{\underline{Y}}_{-1,i}(0) \right) \end{array} \right] - \frac{1}{2} \mu_{c,2} \\
= & \frac{1}{\hat{\sigma}^2} \sum_{i=1}^{n} \left[\begin{array}{c} \left(\Delta_{c_i} \underline{Y}_i - \Delta_{c_i} G_0 \left(\hat{b}_{0i}(c_i) - b_{0i} \right) \right)' \left(\Delta_{c_i} \underline{Y}_i - \Delta_{c_i} G_0 \left(\hat{b}_{0i}(c_i) - b_{0i} \right) \right) \\ - \left(\Delta \underline{Y}_i - \Delta G_0 \left(\hat{b}_{0i}(c_i) - b_{0i} \right) \right)' \left(\Delta \underline{Y}_i - \Delta G_0 \left(\hat{b}_{0i}(c_i) - b_{0i} \right) \right) \end{array} \right] \\
& - \frac{1}{2} \mu_{c,2} \\
= & \frac{1}{\hat{\sigma}^2} V_{fe11,nT}(\mathbb{C}) + \frac{1}{\hat{\sigma}^2} V_{fe12,nT}(\mathbb{C}) - \frac{1}{2} \mu_{c,2},
\end{aligned}$$

where

$$\begin{aligned}
V_{fe11,nT}(\mathbb{C}) &= \sum_{i=1}^{n} \left[(\Delta_{c_i} \underline{Y}_i)' (\Delta_{c_i} \underline{Y}_i) - (\Delta \underline{Y}_i)' (\Delta \underline{Y}_i) \right] \\
&= \frac{1}{n^{1/2}} \sum_{i=1}^{n} c_i \left(\frac{2}{T} \sum_{t=1}^{T} \Delta y_{it} y_{it-1} \right) + \frac{1}{n} \sum_{i=1}^{T} c_i^2 \left(\frac{1}{T^2} \sum_{t=1}^{T} y_{it-1}^2 \right).
\end{aligned}$$

and

$$\begin{aligned}
& V_{fe12,nT}(\mathbb{C}) = \sum_{i=1}^{n} \left[\begin{array}{c} \left(\Delta \underline{Y}_i' \Delta G_0 \right) (\Delta G_0' \Delta G_0)^{-1} (\Delta G_0' \Delta \underline{Y}_i) \\ - \left(\Delta_{c_i} \underline{Y}_i' \Delta_{c_i} G_0 \right) (\Delta_{c_i} G_0' \Delta_{c_i} G_0)^{-1} (\Delta_{c_i} G_0' \Delta_{c_i} \underline{Y}_i) \end{array} \right] \\
= & \sum_{i=1}^{n} \left[y_{i0}^2 - \frac{1}{1 + \frac{c_i^2}{n} \frac{1}{T}} \left(y_{i0} + \frac{c_i}{n^{1/2}} \frac{1}{T} (y_{iT} - y_{i0}) + \frac{c_i^2}{n} \frac{1}{T^2} \sum_{t=1}^{T} y_{it-1} \right)^2 \right].
\end{aligned}$$

Then, we have

$$
\begin{aligned}
& V_{fe1,nT}(\mathbb{C}) \\
= \; & \frac{1}{n^{1/2}} \sum_{i=1}^{n} c_i \left[\left(\frac{2}{T} \sum_{t=1}^{T} \Delta y_{it} y_{it-1} \right) - 2 \left(\frac{y_{i0}}{\sqrt{T}} \right) \left(\frac{y_{iT}}{\sqrt{T}} - \frac{y_{i0}}{\sqrt{T}} \right) \right] \\
& + \frac{1}{n} \sum_{i=1}^{n} c_i^2 \left[\frac{1}{T^2} \sum_{t=1}^{T} y_{it-1}^2 \right] - \frac{1}{2} \mu_{c,2} + O_p\left(\frac{1}{\sqrt{T}} \right).
\end{aligned}
$$

In the Appendix, we show that

$$
\frac{1}{n^{1/2}} \sum_{i=1}^{n} c_i \left(\frac{y_{i0}}{\sqrt{T}} \right) \left(\frac{y_{iT}}{\sqrt{T}} - \frac{y_{i0}}{\sqrt{T}} \right) = O_p\left(\frac{1}{\sqrt{T}} \right). \tag{15}
$$

and so

$$
V_{fe1,nT}(\mathbb{C}) = V_{nT}(\mathbb{C}) + o_p(1).
$$

In view of Theorems 7 and 8 we have the following result.

Theorem 9 *Suppose Assumptions 1 – 6 hold and that $b_{1i}t$ is known. Then, as* $(n, T) \to \infty$

(a) $V_{fe1,nT}(\mathbb{C}) \Rightarrow N\left(-E(c_i\theta_i), 2\mu_{c,2}\right)$.

(b) *The power envelope for invariant testing of H_0 in (3) against H_1 in (4) is $\Phi\left(\sqrt{\frac{\mu_{\theta,2}}{2}} - \bar{z}_\alpha\right)$, where $\mu_{\theta,2} = E\left(\theta_i^2\right)$ and $\bar{z}_\alpha$ is the $(1-\alpha)-$ quantile of the standard normal distribution.*

Remarks

(a) As in the case of $\Psi_{nT}(c)$, we define the test $\Psi_{fe1,nT}(c)$ with a common point $c_i = c$, a constant. Then, the power of the test $\Psi_{fe1,nT}(c)$ is

$$
\Phi\left(\frac{\mu_{\theta,1}}{\sqrt{2}} - \bar{z}_\alpha \right). \tag{16}
$$

(b) With the incidental parameters in the model, Levin *et al* (2002) proposed a panel unit root test based on the pooled OLS estimator. Let $\tilde{z}_{it} = z_{it} - \frac{1}{T}\sum_{t=1}^{T} z_{it}$ and $\tilde{z}_{it} = z_{it-1} - \frac{1}{T}\sum_{t=1}^{T} z_{it-1}$. The t - statistic proposed by Levin *et al* is asymptotically equivalent to the following t - statistic

$$
t^+ = \frac{\sqrt{\sum_{t=1}^{T} \tilde{z}_{it-1}^2} \left(\hat{\rho}_{pool}^+ - 1 \right)}{\frac{\hat{\sigma}}{\sqrt{2}}},
$$

where

$$\hat{\rho}^{+}_{pool} = \frac{\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{z}_{it}\tilde{z}_{it-1} + \frac{nT}{2}\hat{\sigma}^{2}}{\sum_{t=1}^{T}\tilde{z}^{2}_{it-1}}.$$

According to Moon and Perron (2003b), the t^{+} test has significant asymptotic local power within $n^{-1/4}T^{-1}$ neighborhoods of unity. Since $\Psi_{fe1,nT}(c)$ has power in neighborhoods shrinking to unity at the faster rate $n^{-1/2}T^{-1}$, the t^{+} test is inadmissible and asymptotically dominated by $\Psi_{fe1,nT}(c)$.

5 Fixed Effects II: Incidental Trends Case

This section considers the important practical case where the heterogeneous linear trends $b_i' g_t$ are not observable and need to be estimated. We start by considering local neighborhoods of unity that shrink at the rate $\frac{1}{n^{1/4}T}$.

Assumption 10 $\kappa = 1/4$ *in (2).*

We next relax Assumption 6 by allowing that the time series of panel y_{it} can be either stationary or explosive under the alternative hypothesis.

Assumption 11 $\theta_i \sim$ *iid with mean* μ_θ *and variance* σ^2_θ *on a bounded support* $[-M_{l\theta},\, M_{u\theta}]$, *where* $M_{l\theta},\, M_{u\theta} \geq 0$.

Under Assumption 11, we can re-express hypotheses (3) and (4) using the second raw moment of θ_i as follows:

$$\mathbb{H}_0 : \mu_{\theta,2} = 0, \tag{17}$$

and

$$\mathbb{H}_1 : \mu_{\theta,2} > 0. \tag{18}$$

We investigate three panel unit root tests, derive the asymptotic local powers of the tests and compare them.

5.1 Power Envelope

This section derives the power envelope of panel unit root tests for $\mathbb{H}_0$ that are invariant to the transformation $Z \to Z + \beta^* G'$ for arbitrary β^*. Let $\Delta_{\mathbb{C}} Z = (z_0, \Delta_{\mathbb{C}} z_1, ..., \Delta_{\mathbb{C}} z_T)$ and $\Delta_{\mathbb{C}}\beta G' = (\beta g_0, \Delta_{\mathbb{C}}\beta g_1, ..., \Delta_{\mathbb{C}}\beta g_t, ..., \Delta_{\mathbb{C}}\beta g_T)$. Define

$$L_{nT}(\mathbb{C},\beta) = tr(\Delta_{\mathbb{C}} Z - \Delta_{\mathbb{C}}\beta G')(\Delta_{\mathbb{C}} Z - \Delta_{\mathbb{C}}\beta G')'.$$

As above, a (Gaussian) point optimal invariant test statistic can be constructed as:

$$\begin{aligned} V_{fe2,nT}(\mathbb{C}) &= \frac{1}{\hat{\sigma}^2}\left[\min_{\beta} L_{nT}(\mathbb{C},\beta) - \min_{\beta} L_{nT}(0,\beta)\right] \\ &\quad + \left(\frac{1}{n^{1/4}}\sum_{i=1}^{n} c_i\right) + \left(\frac{1}{n^{1/2}}\sum_{i=1}^{n} c_i^2\right)\omega_{p2T} + \left(\frac{1}{n}\sum_{i=1}^{n} c_i^4\right)\omega_{p4T}, \end{aligned}$$

where

$$\omega_{p2T} = -\frac{1}{T}\sum_{t=1}^{T}\frac{t-1}{T} + \frac{2}{T}\sum_{t=1}^{T}\left(\frac{t-1}{T}\right)^2 - \frac{1}{3}$$

$$\omega_{p4T} = \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\frac{t-1}{T}\frac{s-1}{T}\min\left(\frac{t-1}{T},\frac{s-1}{T}\right) - \frac{2}{3}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{t-1}{T}\right)^2 + \frac{1}{9}.$$

For given $c_i's$, the point optimal invariant test, say $\Psi_{fe2,nT}(\mathbb{C})$, rejects the null hypothesis for small values of $V_{fe2,nT}(\mathbb{C})$.

Let $\hat{b}_i(c_i) = (\Delta_{c_i}G'\Delta_{c_i}G)^{-1}(\Delta_{c_i}G'\Delta_{c_i}\underline{Z}_i)$ and $\underline{\hat{Y}}_i(c_i) = \underline{Z}_i - G\hat{b}_i(c_i)' = \underline{Y}_i - G\left(\hat{b}_i(c_i) - b_i\right)'$, and rewrite $V_{fe2,nT}(\mathbb{C})$ as

$$\begin{aligned}
& V_{fe2,nT}(\mathbb{C}) \\
= & \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n}\left(\underline{\hat{Y}}_i(c_i) - \rho_{c_i}\underline{\hat{Y}}_{-1,i}(c_i)\right)'\left(\underline{\hat{Y}}_i(c_i) - \rho_{c_i}\underline{\hat{Y}}_{-1,i}(c_i)\right) \\
& -\frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n}\left(\underline{\hat{Y}}_i(0) - \underline{\hat{Y}}_{-1,i}(0)\right)'\left(\underline{\hat{Y}}_i(0) - \underline{\hat{Y}}_{-1,i}(0)\right) \\
& +\left(\frac{1}{n^{1/4}}\sum_{i=1}^{n}c_i\right) + \left(\frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i^2\right)\omega_{p2T} + \left(\frac{1}{n}\sum_{i=1}^{n}c_i^4\right)\omega_{p4T}.
\end{aligned}$$

In the Appendix, we show that $V_{fe2,nT}(\mathbb{C})$ can be written as

$$\begin{aligned}
& V_{fe2,nT}(\mathbb{C}) \qquad (19) \\
= & \frac{1}{n^{1/4}\hat{\sigma}^2}\sum_{i=1}^{n}c_i\left[\frac{2}{T}\sum_{t=1}^{T}\Delta y_{it}y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \left(\frac{y_{i0}}{\sqrt{T}}\right)^2 + \hat{\sigma}^2\right] \\
& +\frac{1}{n^{1/2}\hat{\sigma}^2}\sum_{i=1}^{n}c_i^2\left[\begin{array}{c}\frac{1}{T^2}\sum_{t=1}^{T}y_{it-1}^2 - 2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right) \\ +\frac{1}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \frac{1}{3}\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}-y_{i0}}{\sqrt{T}}\right) + \hat{\sigma}^2\omega_{p2T}\end{array}\right] \\
& +\frac{1}{n\hat{\sigma}^2}\sum_{i=1}^{n}c_i^4\left[\begin{array}{c}-\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right)^2 + \frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right) \\ -\frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \hat{\sigma}^2\omega_{p4T}\end{array}\right] \\
& +o_p(1)
\end{aligned}$$

when $(n, T \to \infty)$ with $\frac{n^{3/4}}{T} \to 0$.

Lemma 12 *Under Assumptions 1 – 4, 10, and 11, the following hold:*

(a) $\frac{1}{n^{1/4}\hat{\sigma}^2}\sum_{i=1}^{n}c_i\left[\frac{2}{T}\sum_{t=1}^{T}\Delta y_{it}y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \left(\frac{y_{i0}}{\sqrt{T}}\right)^2 + \hat{\sigma}^2\right] = o_p(1);$

(b) $\frac{1}{n^{1/2}\hat{\sigma}^2}\sum_{i=1}^n c_i^2 \left[\begin{array}{c} \left(\frac{1}{T^2}\sum_{t=1}^T y_{it-1}^2 - \hat{\sigma}^2 \frac{1}{T}\sum_{t=1}^T \frac{t-1}{T}\right) + \frac{1}{3}\left\{\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 - \hat{\sigma}^2\right\} \\ -\left\{2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=2}^T \frac{t-1}{T} y_{it-1}\right) - \hat{\sigma}^2 \frac{2}{T}\sum_{t=1}^T \left(\frac{t-1}{T}\right)^2\right\} \\ +\frac{1}{3}\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}-y_{i0}}{\sqrt{T}}\right) \end{array} \right]$

$\Rightarrow N\left(-\frac{1}{90}E\left(c_i^2\theta_i^2\right), \frac{1}{45}E\left(c_i^2\right)\right)$;.

(c) $\frac{1}{n\sigma^2}\sum_{i=1}^n c_i^4 \left[\begin{array}{c} -\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^T \frac{t}{T} y_{it-1}\right)^2 + \frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^T \frac{t}{T} y_{it-1}\right) \\ -\frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \hat{\sigma}^2\omega_{p4T} \end{array} \right] =$

$o_p(1)$.

In view of these results, we have the following theorem.

Theorem 13 *Suppose that Assumptions 1 – 4, 10, and 11. Then,* $V_{fe2,nT}(\mathbb{C}) \Rightarrow N\left(-\frac{1}{90}E\left(c_i^2\theta_i^2\right), \frac{1}{45}E\left(c_i^4\right)\right)$.

From Theorem 13, we find that the size α asymptotic critical value is

$$\psi_{fe2}(\mathbb{C}, \alpha) = -\sqrt{\frac{\mu_{c,4}}{45}}\bar{z}_\alpha,$$

and the asymptotic power is

$$\Phi\left(\frac{1}{6\sqrt{5}}\frac{E\left(c_i^2\theta_i^2\right)}{\sqrt{E\left(c_i^4\right)}} - \bar{z}_\alpha\right). \tag{20}$$

By the Cauchy-Schwarz inequality, we have

$$\Phi\left(\frac{1}{6\sqrt{5}}\frac{E\left(c_i^2\theta_i^2\right)}{\sqrt{E\left(c_i^4\right)}} - \bar{z}_\alpha\right) \leq \Phi\left(\frac{1}{6\sqrt{5}}\sqrt{\mu_{\theta,4}} - \bar{z}_\alpha\right). \tag{21}$$

Again, maximal power, $\Phi\left(\frac{1}{6\sqrt{5}}\sqrt{\mu_{\theta,4}} - \bar{z}_\alpha\right)$, is achieved by choosing $c_i = \theta_i$. According to the Neyman-Pearson lemma, $\Phi\left(\frac{1}{6\sqrt{5}}\sqrt{\mu_{\theta,4}} - \bar{z}_\alpha\right)$ is the power envelope. Summarizing, we have the following theorem.

Theorem 14 *Suppose that the trends* $b_i' g_t$ *in* (1) *are unknown and Assumptions 1 – 4, 10, and 11 hold. Then, the power envelope for testing the null hypothesis* $\mathbb{H}_0$ *in* (3) *against the alternative hypothesis* $\mathbb{H}_1$ *in* (4) *is* $\Phi\left(\frac{1}{6\sqrt{5}}\sqrt{\mu_{\theta,4}} - \bar{z}_\alpha\right)$, *where* $\mu_{\theta,4} = E\left(\theta_i^4\right)$ *and* $\bar{z}_\alpha$ *is the* $(1-\alpha)-$ *quantile of the standard normal distribution.*

Remarks

(a) The power envelope of invariant tests of $\mathbb{H}_0$ in (3) against $\mathbb{H}_1$ depends on the fourth moment of the local to unity parameters $\theta_i' s$.

(b) When the alternative hypothesis is the homogeneous alternative $\mathbb{H}_2$, the power envelope is

$$\Phi\left(\frac{1}{6\sqrt{5}}\theta^2 - \bar{z}_\alpha\right). \tag{22}$$

The power envelope is attained in this case by using $c_i = c$ for any choice of c.

(c) If the θ_i are symmetrically distributed and κ_4 is the 4'the cumulant, then $\sqrt{\mu_{\theta,4}} = \theta^2 \left\{1 + \frac{6\sigma_\theta^2}{\theta^2} + \frac{3\sigma_\theta^4+\kappa_4}{\theta^4}\right\}^{1/2}$ and this will be close to θ^2 when the ratios $\frac{6\sigma_\theta^2}{\theta^2}$ and $\frac{3\sigma_\theta^4+\kappa_4}{\theta^4}$ are small. In such cases, it is clear from (21) that the test with $c_i = c$ for any choice of c will be close to the power envelope.

5.2 Power Comparison

We compare the powers of three tests, which we consider in turn.

5.2.1 The Optimal Invariant Test in Ploberger and Phillips(2002)

The start with the optimal invariant panel unit root test proposed by Ploberger and Phillips (2002). Let $\Delta G' = (g_0, \Delta g_1, ..., \Delta g_T)$ and $\Delta Z = (z_0, \Delta z_1, ..., \Delta z_T)$. Under the null hypothesis, ΔG and ΔZ deliver generalized least squares (GLS) transformations of the trends G and the panel data Z, respectively. To construct the test statistic, we first estimate the trend coefficients β by

$$\bar{\beta} = (\Delta Z \Delta G)\left(\Delta G' \Delta G\right)^{-1},$$

and detrend the panel data Z using this GLS estimate giving

$$E = Z - (\Delta Z \Delta G)\left(\Delta G' \Delta G\right)^{-1} G'.$$

Define

$$V_{g,nT} = \frac{\sqrt{n}}{\hat{\sigma}^2}\left(\frac{1}{nT^2} tr\left(EE'\right) - \hat{\sigma}^2 \omega_{1T}\right), \tag{23}$$

where $\omega_{1T} = \frac{1}{T}\sum_{t=1}^{T} \frac{t}{T}\left(1 - \frac{t}{T}\right)$. In summation notation,

$$V_{g,nT} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{1}{T\hat{\sigma}^2}\sum_{t=1}^{T} \bar{Z}_{it,T}^2 - \omega_{1T}\right], \tag{24}$$

where

$$\bar{Z}_{it,T} = \frac{1}{\sqrt{T}}\left[(z_{it} - z_{i0}) - \frac{t}{T}(z_{iT} - z_{i0})\right],$$

a maximal invariant statistic. In view of (23) and (24), we may interpret $V_{g,nT}$ as the standardized *information* of the GLS detrended panel data. The test $\Psi_{g,nT}$ proposed by Ploberger and Phillips (2002) rejects the null hypothesis $\mathbb{H}_0$ for small values of $V_{g,nT}$.

To investigate the asymptotic power of $\Psi_{g,nT}$, we first derive the asymptotic distribution of $V_{g,nT}$.

Lemma 15 *Suppose Assumptions 1 – 4, 10, and 11 hold. Then,* $V_{g,nT} \Rightarrow N\left(-\frac{1}{90}\mu_{\theta,2}, \frac{1}{45}\right)$.

Using Lemma 15, it is quite straightforward to find the size α asymptotic critical values $\phi_g(\alpha)$ of the test $\Psi_{g,nT}$. For $\bar{z}_\alpha$, the $(1-\alpha)-$ quantile of Z is

$$\phi_g(\alpha) = -\frac{1}{3\sqrt{5}}\bar{z}_\alpha,$$

and the asymptotic local power is

$$\Phi\left(\frac{\mu_{\theta,2}}{6\sqrt{5}} - \bar{z}_\alpha\right), \tag{25}$$

showing that the test $\Psi_{g,nT}$ has significant asymptotic power against the local alternative $\mathbb{H}_1$.

Remarks

(a) Notice that the asymptotic power of the test $\Psi_{g,nT}$ is determined by the second moment of θ_i, $\mu_{\theta,2}$, so that it relies on the variance of θ_i as well as the mean of θ_i.

(b) According to Ploberger and Phillips (2002), the test $\Psi_{g,nT}$ is an optimal invariant test. Let $Q_{\theta,nT}(\theta)$ be the joint probability measure of the data for the given $\theta_i's$ and let v be the probability measure on the space of θ_i. Ploberger and Phillips (2002) show that the test $\Psi_{g,nT}$ is asymptotically the optimal invariant test that maximizes the average power $\int\left(\int \Psi_{g,nT} dQ_{\theta,nT}(\theta)\right) dv$, a quantity which also represents the power of $\Psi_{g,nT}$ against the Bayesian mixture $\int Q_{\theta,nT}(\theta)\, dv$.

(c) Comparing the power (25) of the test $\Psi_{g,nT}$ to the power envelope is straightforward. By the Cauchy-Schwarz inequality we have

$$\Phi\left(\frac{\mu_{\theta,2}}{6\sqrt{5}} - \bar{z}_\alpha\right) \leq \Phi\left(\frac{\sqrt{\mu_{\theta,4}}}{6\sqrt{5}} - \bar{z}_\alpha\right).$$

The test $\Psi_{g,nT}$ achieves the power envelope if the θ_i are constant *a.s.*, that is, the power envelope is achieved against the special alternative hypothesis $\mathbb{H}_2$.

5.2.2 The LM Test in Moon and Phillips(2002)

The second test we investigate is the LM test proposed by Moon and Phillips (2002), which is constructed in a fashion similar to $V_{g,nT}$. The main difference is that Moon and Phillips (2002) use ordinary least squares (OLS) to detrend the data. To fix ideas, define $Q_G = I_T - P_G$ with $P_G = G(G'G)^{-1}G$. Let $D_T = diag(1, T)$. and

$$V_{o,nT} = \frac{\sqrt{n}}{\hat{\sigma}^2}\left(\frac{1}{nT^2}tr(ZQ_GZ') - \hat{\sigma}^2\omega_{2T}\right),$$

where

$$\omega_{2T} = \frac{1}{T}\sum_{t=1}^{T}\frac{t}{T} - \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\frac{\min(t,s)}{T}h_T(t,s),$$

$$h_T(t,s) = g_t' D_T^{-1}\left(\frac{1}{T}\sum_{p=1}^{T} D_T^{-1} g_p g_p' D_T^{-1}\right)^{-1} D_T^{-1} g_s.$$

Define

$$\tilde{Z}_{it,T} = \frac{1}{\sqrt{T}}\left[z_{it} - g_t'\left(\sum_{t=1}^{T} g_t g_t'\right)^{-1}\left(\sum_{t=1}^{T} g_t' z_{it}\right)\right],$$

a scaled version of the OLS detrended panel. Then, we can write

$$V_{o,nT} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T\hat{\sigma}^2}\sum_{t=1}^{T}\tilde{Z}_{it,T}^2 - \omega_{2T}\right],$$

which can be interpreted as the standardized *information* of the detrended panel data. The LM test, say $\Psi_{o,nT}$, of Moon and Phillips (2002) is to reject the null hypothesis $\mathbb{H}_0$ for small values of $V_{o,nT}(c)$.

The following theorem derives the limiting distribution of $V_{o,nT}(c)$.

Lemma 16 *Suppose Assumptions 1 – 4, 10, and 11 hold. Then,* $V_{o,nT} \Rightarrow N\left(-\frac{1}{420}\mu_{\theta,2}, \frac{11}{6300}\right)$.

The size α asymptotic critical value of $\Psi_{o,nT}$, say $\phi_o(\alpha)$, is given by

$$\phi_o(\alpha) = -\sqrt{\frac{11}{6300}}\bar{z}_\alpha,$$

and the asymptotic power is

$$\Phi\left(\frac{\mu_{\theta,2}}{2\sqrt{77}} - \bar{z}_\alpha\right).$$

Remarks

(a) The test $\Psi_{o,nT}$ has significant asymptotic power against the local alternative $\mathbb{H}_1$ and its power depends on the second moment of θ_i, $\mu_{\theta,2}$ just as the power of the test $\Psi_{g,nT}$.

(b) We also find that the asymptotic power of the optimal invariant test $\Psi_{g,nT}$ dominates that of the test $\Psi_{o,nT}$ because $\frac{\mu_\theta^2+\sigma_\theta^2}{2\sqrt{77}} < \frac{\mu_\theta^2+\sigma_\theta^2}{2\sqrt{45}}$. This is perhaps not surprising since the optimal invariant test $\Psi_{g,nT}$ is based on GLS-detrended data, while the test $\Psi_{o,nT}$ is based on OLS-detrended data.

5.2.3 A Common-Point Optimal Invariant Test

As with the test $\Gamma_{nT}(\nleq)$, implementation of the test $V_{p,nT}(\Theta)$ that achieves the power envelope is infeasible. If we use randomly generated $c_i's$ that are independent of θ_i and the panel data z_{it}, according to (20), the power of the test $V_{p,nT}(\mathbb{C})$ is

$$\Phi\left(\frac{1}{6\sqrt{5}}\frac{\mu_{c,2}\mu_{\theta,2}}{\sqrt{\mu_{c,4}}}-\bar{z}_{\alpha}\right). \tag{26}$$

Since $\mu_{c,2} \leq \sqrt{\mu_{c,4}}$, the power (26) is bounded by

$$\Phi\left(\frac{1}{6\sqrt{5}}\mu_{\theta,2}-\bar{z}_{\alpha}\right), \tag{27}$$

which is achieved when we choose $c_i = c$ for $V_{p,nT}(\mathbb{C})$, where c is any positive constant. We denote this test $V_{p,nT}(c)$.

Remarks

(a) The power (27) of the test $V_{fe2,nT}(c)$ is identical to that of the Ploberger and Phillips' optimal invariant test $V_{g,nT}$.

(b) The power of the test $V_{fe2,nT}(c)$ also does not depend on c. It is optimal against the special homogeneous alternative hypothesis $\mathbb{H}_2$ for any choice of c.

(c) As remarked earlier the test $V_{p,nT}(c)$ will achieve power close to the power envelope when the ratios $\frac{6\sigma_\theta^2}{\theta^2}$ and $\frac{3\sigma_\theta^4+\kappa_4}{\theta^4}$ are small.

Remark To simplify analysis, the panel errors u_{it} in model (1) were assumed to be iid across i and t. In empirical applications, we can expect the u_{it} to be serially correlated and possibly heterogeneous across i and sometimes even cross-sectionally dependent. When the u_{it} are cross sectionally independent but not identical and serially correlated, we may replace $\hat{\sigma}^2$ in the test statistics with an estimator of the cross-sectional average of the long-run variances of the u_{it}. An example of such an estimator can be found in Moon and Perron (2003a). When the data are cross section dependent through the presence of some unobservable common factors, one can apply the orthogonalization procedure proposed by Moon and Perron (2003a) and Phillips and Sul (2003) to the panel data after the removal of deterministic components. The tests discussed here may then be constructed using the de-factored data.

6 Simulations

This section reports the results of a small Monte Carlo experiment designed to assess and compare the finite-sample properties of the tests presented earlier in the paper. For this purpose, we use the following data generating process with incidental trends:

$$\begin{aligned} z_{it} &= \alpha_{i0} + \alpha_{i1} t + y_{it} \\ y_{it} &= \rho_i y_{it-1} + u_{it} \\ y_{i0} &= 0 \\ \alpha_{i0}, \alpha_{i1}, u_{it} &\sim N(0,1) \end{aligned}$$

We make two assumptions on the autoregressive parameters:

$$\begin{aligned} \text{Case } A &: \rho_i = 1 \quad \forall i \\ \text{Case } B &: \rho_i \sim U[0.98, 1] \end{aligned}$$

Case A is used to study the size of the proposed tests and Case B to study power. Two features of this case are worth emphasizing. First, we do not impose a common autoregressive parameter under the alternative. As shown above, tests based on pooling are not efficient for this type of alternative. Second, we consider a fixed alternative regardless of the size of n and T to show the increased power as n and/or T increase. Thus, in terms of our theoretical framework, the random variable representing the local alternative, θ_i, has a uniform distribution over the interval $\left[0, \frac{\sqrt{n}T}{50}\right]$. The chosen specification ensures that the average value of ρ_i is 0.99. We consider three values for each of n and $T: n = 10$, 20, and 30 and $T = 100$, 300, and 500. All tests are carried out at the 5% significance level, and the number of replications is set at 1000.

Table 1 presents the results for the size of the tests (case A). Each entry in the table represents the percentage of replications in which the null hypothesis of a unit root is rejected using the appropriate test statistic. The first 3 columns provide results for our point-optimal test for three values of c, that is 1, 2, and 0.5. The next two columns report results for the optimal test of Ploberger and Phillips (2002) and the LM test of Moon and Phillips (2002). Finally, the last column reports results for the t-ratio type test as in Moon and Perron (2003a).

The size properties of the point-optimal test appear quite sensitive to the choice of c and values of n and T. If we fix T and c, there is an increasing relation between the rejection probability and n. Similarly, for fixed n and c, there is a decreasing relation between the rejection probability and T. Finally, for fixed n and T, larger c implies a lower rejection probability. It is therefore difficult to come up with a good choice of c based on these results, although values between 1 and 2 provide a good trade-off for all values of n and T.

The next two columns show that both the Ploberger-Phillips and Moon-Phillips tests tend to underreject, sometimes quite severely. Finally, the t-type test seems to have best size overall.

Table 2 presents the more interesting power results for our experiment. All figures in the table are rejection probabilities for case B where ρ_i has a uniform distribution between 0.98 and 1 based on the 5% empirical critical values. In this case, the choice of c is not important at all, as predicted by asymptotic theory. All rejection probabilities are close to one another for the three choices of c. In fact, the variations are all within 2 simulation standard deviations (given by $\sqrt{\frac{p(1-p)}{1000}}$), and all differences are probably due to experimental randomness. The Ploberger-Phillips test also behaves in a very similar way, as predicted by the asymptotics.

The LM test of Moon and Phillips has good power but appears to be slightly dominated by the other two tests, as again predicted by the theory. Finally, the t-type test has no power beyond size as shown by Moon and Perron (2003a).

7 Conclusion

In terms of their asymptotic power functions, the pooled t- test, the Ploberger-Phillips (2002) test and the point optimal test all have good discriminatory power against a unit root null in shrinking neighborhoods of unity. When the alternative is homogeneous it is possible to attain the asymptotic power envelope and both the Ploberger- Phillips test and the point optimal test are uniformly most powerful in this case. Interestingly, the point optimal test has this property irrespective of the common alternative point chosen to set up the test. This is in contrast to point optimal tests of a unit root that are based solely on time series data (Elliot et. al. 1996), where no test is uniformly most powerful and an arbitrary selection of a common point is needed in the construction of the test.

An important empirical consequence of the present investigation is that increasing the complexity of the fixed effects in a panel model inevitably reduces the potential power of unit root tests. This reduction in power has a quantitative manifestation in the radial order of the shrinking neighborhoods around unity for which asymptotic power is non negligible. When there are no fixed effects or constant fixed effects, tests have power in a neighborhood of unity of order $n^{-1/2}T^{-1}$. When incidental trends are fitted, the tests only have power in a larger neighborhood of order $n^{-1/4}T^{-1}$. A continuing reduction in power is to be expected as higher order incidental trends are fitted in a panel model. The situation is analogous to what happens in time series models where unit root nonstationary data is fitted by a lagged variable and deterministic trends. In such cases, both the lagged variable and the deterministic trends compete to model the nonstationarity in the data with the upshot that the rate of convergence is affected. In particular, Phillips (2002) showed that rate of convergence to a unit root is slowed by the presence of increasing numbers of deterministic regressors. In the panel model context, the present paper shows that discriminatory power against a unit root is weakened as more complex deterministic regressors are included in the panel model.

8 Appendix: Technical Proofs

We let $z_{it}(0)$ and $y_{it}(0)$, respectively, denote the panel observations z_{it} and y_{it} that are generated by model (1) with $\rho_i = 1$, that is, $\theta_i = 0$. We also define $Z(0), Y(0), Y_{-1}(0)$, respectively, in similar fashion from Z, Y, and Y_{-1}. Also, for notational simplicity, we write $u_{i1} = y_{i1}$. Finally, define

$$h(r,s) = (1,r)\begin{pmatrix} 1 & \int_0^1 rdr \\ \int_0^1 rdr & \int_0^1 r^2dr \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ s \end{pmatrix} = 4 - 6r - 6s + 12rs.$$

8.1 Preliminary Results

First, we introduce a lemma that is useful in the proof of the main results. Suppose that c_i are sequence of iid random variables whose supports are the same of those of $\theta_i's$ and are independent of u_{it} for all i and t.

Lemma 17 *Suppose that Assumptions 1 – 4, 10, and 11 hold. Then, the following hold as* $(n, T \to \infty)$ *with* $\frac{\sqrt{n}}{T} \to 0$.

(a) $\frac{1}{\sqrt{n}}\sum_{i=1}^n c_i^2\left[\frac{1}{T^2\sigma^2}\sum_{t=1}^T\left\{(y_{it}-y_{i0}) - \frac{t}{T}(y_{iT}-y_{i0})\right\}^2 - \omega_{1T}\right] \Rightarrow N\left(-\frac{E\left(c_i^2\theta_i^2\right)}{90}, \frac{E\left(c_i^4\right)}{45}\right)$

(b) $\frac{1}{\sqrt{n}\sigma^2}\sum_{i=1}^n\left[\frac{1}{T^2}\sum_{t=1}^T y_{it}^2 - \frac{1}{T^3}\sum_{t=1}^T\sum_{s=1}^T y_{it}y_{is}h_T(t,s) - \omega_{2T}\right] \Rightarrow N\left(-\frac{E\left(\theta_i^2\right)}{420}, \frac{11}{6300}\right)$.

Proof of Lemma 17
Part (a): For notational simplicity let $\bar{Y}_{it,T} = (y_{it}-y_{i0}) - \frac{t}{T}(y_{iT}-y_{i0})$ and $\bar{Y}_{it,T}(0) = (y_{it}(0)-y_{i0}(0)) - \frac{t}{T}(y_{iT}(0)-y_{i0}(0))$. Using this notation, we decompose

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^n c_i^2\left[\frac{1}{T^2\sigma^2}\sum_{t=1}^T\left\{(y_{it}-y_{i0}) - \frac{t}{T}(y_{iT}-y_{i0})\right\}^2 - \omega_{1T}\right] \\
=\ & \frac{1}{\sqrt{n}}\sum_{i=1}^n c_i^2\left(\frac{1}{T^2\sigma^2}\sum_{t=1}^T \bar{Y}_{it,T}^2(0) - \omega_{1T}\right) + \frac{1}{\sqrt{n}}\sum_{i=1}^n c_i^2\left(\frac{1}{T^2\sigma^2}\sum_{t=1}^T\left(\bar{Y}_{it,T} - \bar{Y}_{it,T}(0)\right)^2\right) \\
& + \frac{2}{\sqrt{n}}\sum_{i=1}^n c_i^2\left(\frac{1}{T^2\sigma^2}\sum_{t=1}^T \bar{Y}_{it,T}(0)\left(\bar{Y}_{it,T} - \bar{Y}_{it,T}(0)\right)\right) \\
=\ & I_a + II_a + III_a, \text{ say.}
\end{aligned}$$

Notice by a direct calculation that

$$E\left[c_i^2\left(\frac{1}{T^2\sigma^2}\sum_{t=1}^T \bar{Y}_{it,T}^2(0) - \omega_{2T}\right)\right] = O\left(\frac{1}{T}\right).$$

Since $\frac{\sqrt{n}}{T} \to 0$, by applying Theorem 3 in Phillips and Moon (1999), we have

$$I_a \Rightarrow N\left(0, \frac{1}{45}E\left(c_i^4\right)\right). \tag{28}$$

For term II_a, by definition we have

$$\begin{aligned}
& II_a \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2\left(\frac{1}{T^2\sigma^2}\sum_{t=1}^{T}\left(y_{it}-y_{it}(0)\right)^2\right) \\
& -\frac{2}{\sqrt{n}}\sum_{i=1}^{n} c_i^2\left(\frac{1}{T^2\sigma^2}\sum_{t=1}^{T}\left(\frac{t}{T}\right)\left(y_{it}-y_{it}(0)\right)\left(y_{iT}-y_{iT}(0)\right)\right) \\
& +\frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2\left(\frac{1}{T^2\sigma^2}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^2\left(y_{iT}-y_{iT}(0)\right)^2\right) \\
= & II_{a1}+II_{a2}+II_{a3}, \text{ say.}
\end{aligned}$$

Notice by definition that

$$\begin{aligned}
y_{it}-y_{it}(0) &= \sum_{p=0}^{t-1}\left(\rho_i^{t-p}-1\right)u_{ip}=\sum_{p=0}^{t-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{\kappa}T}\right)^{l}\right]u_{ip} \text{ for } t\geq 1 \\
&= 0 \text{ for } t=0, \qquad (29)
\end{aligned}$$

where we set $u_{i0}=y_{i0}$ for notational convenience. Recall that $\kappa=\frac{1}{4}$. By (29) and applying Corollary 1 in Phillips and Moon (1999), we have

$$\begin{aligned}
II_{a1} &\sim \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2\theta_i^2\frac{1}{T\sigma^2}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t-1}\left(\frac{t-p}{T}\right)u_{ip}\right)^2 \\
&\to_p E\left(c_i^2\theta_i^2\right)\int_0^1\int_0^r (r-s)^2\,dsdr=\frac{1}{12}E\left(c_i^2\theta_i^2\right),
\end{aligned}$$

$$\begin{aligned}
II_{a2} &\sim -\frac{2}{\sqrt{n}}\sum_{i=1}^{n} c_i^2\theta_i^2\left(\frac{1}{T\sigma^2}\sum_{t=1}^{T}\left(\frac{t-1}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t-1}\left(\frac{t-p}{T}\right)u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-1}\left(\frac{T-q}{T}\right)u_{iq}\right)\right) \\
&\to_p -2E\left(c_i^2\theta_i^2\right)\int_0^1 r\int_0^r (r-s)(1-s)\,dsdr=-\frac{11}{60}E\left(c_i^2\theta_i^2\right),
\end{aligned}$$

and

$$\begin{aligned}
II_{a3} &\sim \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2\theta_i^2\frac{1}{T\sigma^2}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^2\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T-1}\left(\frac{T-p}{T}\right)u_{ip}\right)^2 \\
&\to_p E\left(c_i^2\theta_i^2\right)\int_0^1 r^2dr\int_0^1 (1-r)^2\,dr=\frac{1}{9}E\left(c_i^2\theta_i^2\right).
\end{aligned}$$

Combining the limits of II_{a1}, II_{a2}, and II_{a3}, we have

$$I_2 \to_p \frac{1}{90}E\left(c_i^2\theta_i^2\right). \qquad (30)$$

Next, for III_a, write $X_{iT} = \frac{1}{T^2\sigma^2}\sum_{t=1}^{T}\bar{Y}_{it,T}(0)\left(\bar{Y}_{it,T} - \bar{Y}_{it,T}(0)\right)$. Also define

$$X_{1iT} = \frac{1}{T\sigma^2}\sum_{t=1}^{T}\left[\begin{array}{c} \left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t}u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-1}\left(\frac{t-q}{T}\right)u_{iq}\right) \\ -\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t}u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-1}\left(\frac{T-q}{T}\right)u_{iq}\right) \\ -\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T}u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-1}\left(\frac{t-q}{T}\right)u_{iq}\right) \\ +\left(\frac{t}{T}\right)^2\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T}u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-1}\left(\frac{T-q}{T}\right)u_{iq}\right) \end{array}\right],$$

and

$$X_{2iT} = \frac{1}{T\sigma^2}\sum_{t=1}^{T}\left[\begin{array}{c} \left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t}u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-2}\left(\frac{t-q}{T}\right)\left(\frac{t-q-1}{T}\right)u_{iq}\right) \\ -\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t}u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-2}\left(\frac{T-q}{T}\right)\left(\frac{T-q-1}{T}\right)u_{iq}\right) \\ -\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T}u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-2}\left(\frac{t-q}{T}\right)\left(\frac{t-q-1}{T}\right)u_{iq}\right) \\ +\left(\frac{t}{T}\right)^2\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T}u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-2}\left(\frac{T-q}{T}\right)\left(\frac{T-q-1}{T}\right)u_{iq}\right) \end{array}\right].$$

Then, by (29), we have

$$III_a \sim -\frac{2}{n^{3/4}}\sum_{i=1}^{n}c_i^2\theta_i X_{1iT} + \frac{1}{n}\sum_{i=1}^{n}c_i^2\theta_i^2 X_{2iT} = -2III_{a1} + III_{a2}, \text{ say.}$$

A direct calculation shows that

$$\begin{aligned} & EIII_{a1} \\ = \; & \frac{E\left(c_i^2\theta_i\right)}{n^{3/4}}\sum_{i=1}^{n}EX_{1iT} \\ = \; & E\left(c_i^2\theta_i\right)n^{1/4}\frac{1}{T}\sum_{t=1}^{T}\left[\begin{array}{c} \frac{1}{T}\sum_{p=0}^{t-1}\frac{t-p}{T} - \left(\frac{t}{T}\right)\frac{1}{T}\sum_{p=0}^{t-1}\left(\frac{T-p}{T}\right) \\ -\left(\frac{t}{T}\right)\frac{1}{T}\sum_{p=0}^{t-1}\left(\frac{t-p}{T}\right) + \left(\frac{t}{T}\right)^2\frac{1}{T}\sum_{p=0}^{T-1}\left(\frac{T-p}{T}\right) \end{array}\right] \\ = \; & E\left(c_i^2\theta_i\right)n^{1/4}\int_0^1\left(\int_0^r (r-s)\,ds - r\int_0^r (1-s)\,ds - r\int_0^r (r-s)\,ds + r^2\int_0^1 (1-s)\,ds\right)dr \\ & +O\left(\frac{n^{1/4}}{T}\right) \\ = \; & o(1), \end{aligned}$$

since $\int_0^1 \left(\int_0^r (r-s)\, ds - r \int_0^r (1-s)\, ds - r \int_0^r (r-s)\, ds + r^2 \int_0^1 (1-s)\, ds \right) dr =$ 0 and $\frac{n^{1/4}}{T} \to 0$ by Assumption 3. Also,

$$
\begin{aligned}
& E\left(c_i^4 \theta_i^2 X_{1iT}^2\right) \\
\leq \quad & \frac{2E\left(c_i^4 \theta_i^2\right)}{T\sigma^2} \sum_{t=1}^{T} \left\{ \begin{array}{c}
E\left[\left(\frac{1}{\sqrt{T}} \sum_{p=0}^{t} u_{ip}\right)\left(\frac{1}{\sqrt{T}} \sum_{q=0}^{t-1} \left(\frac{t-q}{T}\right) u_{iq}\right)\right]^2 \\
+E\left[\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}} \sum_{p=0}^{t} u_{ip}\right)\left(\frac{1}{\sqrt{T}} \sum_{q=0}^{T-1} \left(\frac{T-q}{T}\right) u_{iq}\right)\right]^2 \\
+E\left[\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}} \sum_{p=0}^{T} u_{ip}\right)\left(\frac{1}{\sqrt{T}} \sum_{q=0}^{t-1} \left(\frac{t-q}{T}\right) u_{iq}\right)\right]^2 \\
+E\left[\left(\frac{t}{T}\right)^2\left(\frac{1}{\sqrt{T}} \sum_{p=0}^{T} u_{ip}\right)\left(\frac{1}{\sqrt{T}} \sum_{q=0}^{T-1} \left(\frac{T-q}{T}\right) u_{iq}\right)\right]^2
\end{array} \right\} \\
= \quad & M \text{ for some finite constant } M.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E\left(III_{a1}^2\right) &= Var\left(III_{a1}\right) + \left(E\left(III_{a1}\right)\right)^2 \\
&\leq \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} E\left(c_i^2 \theta_i\right) E\left(X_{i1T}^2\right) + \left(EI_3\right)^2 \\
&= O\left(\frac{1}{n^{1/2}}\right) + O\left(\frac{n^{1/2}}{T^2}\right) = o(1),
\end{aligned}
$$

which yields

$$III_{a1} = o_p(1).$$

Next, applying Corollary 1 in Phillips and Moon (1999), we have

$$
\begin{aligned}
III_{a2} \quad & \to_p \quad E\left(c_i^2 \theta_i^2\right) \left[\begin{array}{c}
\int_0^1 \int_0^r (r-s)^2\, dsdr - \int_0^1 r \int_0^r (1-s)^2\, dsdr \\
- \int_0^1 r \int_0^r (r-s)^2\, dsdr + \int_0^1 r^2 dr \left(\int_0^1 (1-s)^2\, ds \right)
\end{array} \right] \\
& = \quad -\frac{1}{45} E\left(c_i^2 \theta_i^2\right).
\end{aligned}
$$

Combining the limits of I_{31} and I_{32}, we have

$$III_a \to_p -\frac{1}{45} E\left(c_i^2 \theta_i^2\right). \tag{31}$$

From (28), (30), and (31), we have the required result for Part (a). ■

Part (b): In matrix notation write

$$
\begin{aligned}
& \frac{1}{\sqrt{n}\sigma^2}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}y_{it}^2-\frac{1}{T^3}\sum_{t=1}^{T}\sum_{s=1}^{T}y_{it}y_{is}h_T(t,s)-\omega_{2T}\right] \\
= & \sqrt{n}\left(\frac{1}{nT^2\sigma^2}tr\left(YQ_GY'\right)-\omega_{2T}\right) \\
= & \sqrt{n}\left(\frac{1}{nT^2\sigma^2}tr\left(Y(0)Q_GY(0)'\right)-\omega_{2T}\right) \\
& +\sqrt{n}\left(\frac{1}{nT^2\sigma^2}tr\left(YQ_GY'-Y(0)Q_GY(0)'\right)\right) \\
= & I_b+II_b, \text{ say.}
\end{aligned}
$$

Rewriting the term I_b in summation notation,

$$
I_b=\frac{1}{\sqrt{n}\sigma^2}\sum_{i=1}^{n}\left\{\frac{1}{T^2}\sum_{t=1}^{T}y_{it}(0)^2-\frac{1}{T^3}\sum_{t=1}^{T}\sum_{s=1}^{T}y_{it}(0)\,y_{is}(0)\,h_T(t,s)-\omega_{2T}\right\},
$$

and noticing that

$$
E\left(\frac{1}{T^2}\sum_{t=1}^{T}y_{it}(0)^2-\frac{1}{T^3}\sum_{t=1}^{T}\sum_{s=1}^{T}y_{it}(0)\,y_{is}(0)\,h_T(t,s)\right)=\sigma^2\omega_{2T},
$$

we apply Theorem 3 in Phillips and Moon (1999) and deduce that

$$
I_b\Rightarrow N\left(0,\frac{11}{6300}\right). \tag{32}
$$

For II_b, we further decompose the term II_b into

$$
\begin{aligned}
II_b &= \frac{1}{\sqrt{n}T^2\sigma^2}tr\left[(Y-Y(0))\,Q_G\,(Y-Y(0))'\right]+\frac{2}{\sqrt{n}T^2\sigma^2}tr\left[(Y-Y(0))\,Q_GY(0)'\right] \\
&= II_{b1}+II_{b2}, \text{ say.}
\end{aligned}
$$

Write

$$
II_{b1}=\frac{1}{\sqrt{n}T^2\sigma^2}\sum_{i=1}^{n}\sum_{t=1}^{T}(y_{it}-y_{it}(0))^2-\frac{1}{\sqrt{n}T^3\sigma^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}(y_{it}-y_{it}(0))\,(y_{is}-y_{is}(0))\,h_T(t,s).
$$

Then, by (29) and applying Corollary 1 in Phillips and Moon (1999), we have

$$
\begin{aligned}
& \frac{1}{\sqrt{n}T^2\sigma^2}\sum_{i=1}^{n}\sum_{t=1}^{T}(y_{it}-y_{it}(0))^2 \\
= & \frac{1}{n}\sum_{i=1}^{n}\theta_i^2\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(\sum_{p=0}^{t-1}\left(\frac{t-p}{T}\right)u_{ip}\right)^2\right]+O_p\left(\frac{1}{n^{1/4}}\right) \\
\to_p & E\left(\theta_i^2\right)\int_0^1\int_0^r(r-s)^2\,dsdr=\frac{1}{12}E\left(\theta_i^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{\sqrt{n}T^3\sigma^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(y_{it}-y_{it}\left(0\right)\right)\left(y_{is}-y_{is}\left(0\right)\right)h_T\left(t,s\right) \\
= & \frac{1}{n}\sum_{i=1}^{n}\theta_i^2\frac{1}{T^3}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{p=0}^{t-1}\sum_{q=0}^{s-1}\left(\frac{t-p}{T}\right)\left(\frac{s-q}{T}\right)h_T\left(t,s\right)u_{ip}u_{iq} \\
& +O_p\left(\frac{1}{n^{1/4}}\right) \\
\to_p & \ E\left(\theta_i^2\right)\int_0^1\int_0^1\int_0^{r\wedge s}\left(r-p\right)\left(s-p\right)h\left(r,s\right)dpdsdr=\frac{17}{210}E\left(\theta_i^2\right).
\end{aligned}
$$

Therefore,

$$
II_{b1}\to_p\frac{1}{420}E\left(\theta_i^2\right). \tag{33}
$$

Next, in view of (29) with $\kappa=\frac{1}{4}$, we may have

$$
II_{b2}=-\frac{2}{n^{3/4}\sigma^2}\sum_{i=1}^{n}\theta_iX_{1iT}+\frac{1}{n\sigma^2}\sum_{i=1}^{n}\theta_i^2X_{2iT}+o_p\left(1\right),
$$

where

$$
X_{1iT}=\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=0}^{t-1}\sum_{q=0}^{t}\left(\frac{t-s}{T}\right)u_{is}u_{iq}-\frac{1}{T^3}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{s=0}^{t-1}\sum_{q=0}^{p}\left(\frac{t-s}{T}\right)h_T\left(t,p\right)u_{is}u_{iq},
$$

and

$$
\begin{aligned}
X_{2iT} = & \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=0}^{t-1}\sum_{q=0}^{t}\left(\frac{t-s}{T}\right)\left(\frac{t-s-1}{T}\right)u_{is}u_{iq} \\
& -\frac{1}{T^3}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{s=0}^{t-1}\sum_{q=0}^{p}\left(\frac{t-s}{T}\right)\left(\frac{t-s-1}{T}\right)h_T\left(t,p\right)u_{is}u_{iq}.
\end{aligned}
$$

A direct calculation shows that

$$
EX_{1iT}=\sigma^2\left[\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)-\frac{1}{T^3}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{s=0}^{t\wedge p-1}\left(\frac{t-s}{T}\right)h_T\left(t,p\right)\right]=O\left(\frac{1}{T}\right),
$$

because

$$
EX_{1iT}-\sigma^2\int_0^1\int_0^r\left(r-s\right)dsdr+\sigma^2\int_0^1\int_0^1\int_0^{r\wedge p}\left(r-s\right)h\left(r,p\right)dsdpdr=O\left(\frac{1}{T}\right),
$$

and

$$
\int_0^1\int_0^r\left(r-s\right)dsdr-\int_0^1\int_0^1\int_0^{r\wedge p}\left(r-s\right)h\left(r,p\right)dsdpdr=0.
$$

Also,

$$\begin{aligned}
& EX_{1iT}^2 \\
\leq\ & \frac{2}{T^4}\sum_{t=1}^{T}\sum_{x=1}^{T}\sum_{s=0}^{t-1}\sum_{y=0}^{x-1}\sum_{q=0}^{t}\sum_{z=0}^{x}\left(\frac{t-s}{T}\right)\left(\frac{x-y}{T}\right)E\left[u_{is}u_{iq}u_{iy}u_{iz}\right] \\
& +\frac{2}{T^6}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{x=1}^{T}\sum_{y=1}^{T}\sum_{s=0}^{t-1}\sum_{q=0}^{p-1}\sum_{z=0}^{x}\sum_{w=0}^{y}\left(\frac{t-s}{T}\right)\left(\frac{x-z}{T}\right)h_T\left(t,p\right)h_T\left(x,y\right)E\left[u_{is}u_{iq}u_{iz}u_{iw}\right] \\
=\ & O\left(1\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
-\frac{1}{n^{3/4}}\sum_{i=1}^{n}\theta_i X_{1iT} &= -\frac{1}{n^{3/4}}\sum_{i=1}^{n}\theta_i\left(X_{1iT}-EX_{1iT}\right)+\frac{1}{n^{3/4}}\sum_{i=1}^{n}\theta_i\left(EX_{1iT}\right) \\
&= O_p\left(\frac{1}{n^{1/4}}\right)+O\left(\frac{n^{1/4}}{T}\right)=o_p\left(1\right).
\end{aligned}$$

Next, by Corollary 1 in Phillips and Moon (1999), we have

$$\begin{aligned}
\frac{1}{n\sigma^2}\sum_{i=1}^{n}\theta_i^2 X_{2iT} &\to_p E\left(\theta_i^2\right)\left[\int_0^1\int_0^r\left(r-s\right)^2 dsdr-\int_0^1\int_0^1\int_0^{r\wedge p}\left(r-s\right)^2 h\left(r,p\right)dsdpdr\right] \\
&= -E\left(\theta_i^2\right)\frac{1}{210}.
\end{aligned}$$

Therefore, we have

$$II_{b2}\to_p -E\left(\theta_i^2\right)\frac{1}{210}. \tag{34}$$

Combining the limits of the terms I_b, II_{b1}, and II_{b2} in (32), (33), and (34), respectively, we have the desired result for Part (b). ■

8.2 Proofs and Derivations of the Main Results

Proof of (15).

Split the term (15) as

$$\begin{aligned}
& \frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}}{\sqrt{T}}-\frac{y_{i0}}{\sqrt{T}}\right) \\
=\ & \frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}\left(0\right)}{\sqrt{T}}-\frac{y_{i0}}{\sqrt{T}}\right)+\frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}}{\sqrt{T}}-\frac{y_{iT}\left(0\right)}{\sqrt{T}}\right).
\end{aligned}$$

Notice that the first term is

$$\frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}\left(0\right)}{\sqrt{T}}-\frac{y_{i0}}{\sqrt{T}}\right)=\frac{1}{\sqrt{T}}\left\{\frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i y_{i0}\left(\frac{1}{T^{1/2}}\sum_{t=1}^{T}u_{it}\right)\right\}=O_p\left(\frac{1}{\sqrt{T}}\right).$$

Next, from (29) we have

$$\frac{y_{iT}}{\sqrt{T}}-\frac{y_{iT}(0)}{\sqrt{T}}=\frac{1}{T^{1/2}}\sum_{p=0}^{T-1}\left(\rho_i^{T-p}-1\right)u_{ip}=\frac{1}{T^{1/2}}\sum_{p=0}^{T-1}\left[\sum_{l=1}^{T-p}\binom{T-p}{l}\left(\frac{-\theta_i}{n^{1/2}T}\right)^l\right]u_{ip}.$$

Then, the second term is

$$\begin{aligned}
&\frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}}{\sqrt{T}}-\frac{y_{iT}(0)}{\sqrt{T}}\right)\\
=&\ -\frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}c_i\theta_i y_{i0}\left(\frac{1}{T^{1/2}}\sum_{p=0}^{T-1}\frac{T-p}{T}u_{ip}\right)+o_p\left(\frac{1}{\sqrt{T}}\right)=O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}$$

as required. ■

Derivation of $V_{fe2,nT}(\mathbb{C})$ **in** (19).

By definition, we write

$$\begin{aligned}
&V_{fe2,nT}(\mathbb{C})\\
=&\ \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\hat{\underline{Y}}_i(c_i)-\rho_{c_i}\hat{\underline{Y}}_{-1,i}(c_i)\right)'\left(\hat{\underline{Y}}_i(c_i)-\rho_{c_i}\hat{\underline{Y}}_{-1,i}(c_i)\right)\\-\left(\hat{\underline{Y}}_i(0)-\hat{\underline{Y}}_{-1,i}(0)\right)'\left(\hat{\underline{Y}}_i(0)-\hat{\underline{Y}}_{-1,i}(0)\right)\end{array}\right]\\
&+\left(\frac{1}{n^{1/4}}\sum_{i=1}^{n}c_i\right)+\left(\frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i^2\right)\omega_{p2T}+\left(\frac{1}{n}\sum_{i=1}^{n}c_i^4\right)\omega_{p4T}\\
=&\ \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\Delta_{c_i}\underline{Y}_i-\Delta_{c_i}G\left(\hat{\beta}_i(c_i)-\beta_i\right)\right)'\left(\Delta_{c_i}\underline{Y}_i-\Delta_{c_i}G\left(\hat{\beta}_i(c_i)-\beta_i\right)\right)\\-\left(\Delta\underline{Y}_i-\Delta G\left(\hat{\beta}_i(0)-\beta_i\right)\right)'\left(\Delta\underline{Y}_i-\Delta G\left(\hat{\beta}_i(0)-\beta_i\right)\right)\end{array}\right]\\
&+\left(\frac{1}{n^{1/4}}\sum_{i=1}^{n}c_i\right)+\left(\frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i^2\right)\omega_{p2T}+\left(\frac{1}{n}\sum_{i=1}^{n}c_i^4\right)\omega_{p4T}\\
=&\ \frac{1}{\hat{\sigma}^2}V_{fe21,nT}(\mathbb{C})+\frac{1}{\hat{\sigma}^2}V_{fe22,nT}(\mathbb{C})\\
&+\left(\frac{1}{n^{1/4}}\sum_{i=1}^{n}c_i\right)+\left(\frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i^2\right)\left(-\frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{T}+2\frac{1}{T}\sum_{t=2}^{T}\left(\frac{t-1}{T}\right)^2-\frac{1}{3}\right)\\
&+\left(\frac{1}{n}\sum_{i=1}^{n}c_i^4\right)\left(\frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=2}^{T}\frac{t-1}{T}\frac{s-1}{T}\min\left(\frac{t-1}{T},\frac{s-1}{T}\right)-\frac{2}{3}\left(\frac{1}{T}\sum_{t=2}^{T}\left(\frac{t-1}{T}\right)^2\right)+\frac{1}{9}\right),\\
&\text{say,}
\end{aligned}$$

where

$$\begin{aligned}
V_{fe21,nT}(\mathbb{C}) &= \sum_{i=1}^{n}\left[\left(\Delta_{c_i}\underline{Y}_i\right)'\left(\Delta_{c_i}\underline{Y}_i\right)-\left(\Delta\underline{Y}_i\right)'\left(\Delta\underline{Y}_i\right)\right]\\
V_{fe22,nT}(\mathbb{C}) &= \sum_{i=1}^{n}\left[\left(\Delta\underline{Y}_i\right)'\Delta G\left(\Delta G'\Delta G\right)^{-1}\Delta G'\left(\Delta\underline{Y}_i\right)-\left(\Delta_{c_i}\underline{Y}_i\right)'\Delta_{c_i}G\left(\Delta_{c_i}G'\Delta_{c_i}G\right)^{-1}\Delta_{c_i}G'\left(\Delta_{c_i}\underline{Y}_i\right)\right]
\end{aligned}$$

By definition,

$$V_{fe21,nT}(\mathbb{C}) = \frac{2}{n^{1/4}} \sum_{i=1}^{n} c_i \left(\frac{1}{T} \sum_{t=1}^{T} \Delta y_{it} y_{it-1} \right) + \frac{1}{n^{1/2}T^2} \sum_{i=1}^{T} c_i^2 \sum_{t=1}^{T} y_{it-1}^2.$$

Next, denoting $D = diag(\sqrt{T}, 1)$ and $\tilde{G} = GD$, we rewrite

$$\begin{aligned}
& V_{fe22,nT}(\mathbb{C}) \\
= & \sum_{i=1}^{n} \left(\frac{1}{\sqrt{T}} (\Delta \underline{Y}_i)' \Delta \tilde{G} \right) \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} \left(\frac{1}{\sqrt{T}} \Delta \tilde{G}' (\Delta \underline{Y}_i) \right) \\
& - \sum_{i=1}^{n} \left(\frac{1}{\sqrt{T}} (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} \tilde{G} \right) \left(\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} \right)^{-1} \left(\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) \right) \\
= & \sum_{i=1}^{n} tr \left[\begin{array}{c} \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} \\ \times \left\{ \left(\frac{1}{\sqrt{T}} \Delta \tilde{G}' (\Delta \underline{Y}_i) \right) \left(\frac{1}{\sqrt{T}} (\Delta \underline{Y}_i)' \Delta \tilde{G} \right)' - \left(\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) \right) \left(\frac{1}{\sqrt{T}} (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} \tilde{G} \right)' \right\} \end{array} \right] \\
& + \sum_{i=1}^{n} tr \left[\left\{ \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} - \left(\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} \right)^{-1} \right\} \left(\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) \right) \left(\frac{1}{\sqrt{T}} (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} \tilde{G} \right)' \right] \\
= & V_{fe221,nT}(\mathbb{C}) + V_{fe222,nT}(\mathbb{C}), \text{ say.}
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} &= \begin{pmatrix} 1 + \frac{c_i^2}{n^{1/2}} \frac{1}{T} & \frac{1}{\sqrt{T}} \left(\frac{c_i}{n^{1/4}} + \frac{c_i^2}{n^{1/2}} \left(\frac{1}{T} \sum_{t=1}^{T} \frac{t}{T} \right) \right) \\ \frac{1}{\sqrt{T}} \left(\frac{c_i}{n^{1/4}} + \frac{c_i^2}{n^{1/2}} \left(\frac{1}{T} \sum_{t=1}^{T} \frac{t}{T} \right) \right) & \frac{1}{T} \sum_{t=1}^{T} \left(1 + \frac{c_i}{n^{1/4}} \frac{t}{T} \right)^2 \end{pmatrix}, \\
\frac{1}{\sqrt{T}} \Delta \tilde{G}' (\Delta \underline{Y}_i) &= \begin{pmatrix} y_{i0} \\ \frac{1}{\sqrt{T}} (y_{iT} - y_{i0}) \end{pmatrix}, \\
\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) &= \begin{pmatrix} y_{i0} + \frac{c_i}{n^{1/4}} \frac{1}{T} (y_{iT} - y_{i1}) + \frac{c_i^2}{n^{1/2}} \frac{1}{T^2} \sum_{t=1}^{T} y_{it-1} \\ \frac{1}{\sqrt{T}} (y_{iT} - y_{i0}) + \frac{c_i}{n^{1/4}} \frac{1}{\sqrt{T}} y_{iT} + \frac{c_i^2}{n^{1/2}} \frac{1}{T\sqrt{T}} \sum_{t=1}^{T} \frac{t}{T} y_{it-1} \end{pmatrix}.
\end{aligned}$$

Computation of $V_{fe221,nT}(\mathbb{C})$: A direct calculation shows that

$$\begin{aligned}
& V_{fe221,nT}(\mathbb{C}) \\
= & \frac{1}{n^{1/4}}\sum_{i=1}^{n} c_i\left(2\left(\frac{y_{i0}}{\sqrt{T}}\right)^2 - 2\left(\frac{y_{iT}}{\sqrt{T}}\right)^2\right) \\
& +\frac{1}{n^{1/2}}\sum_{i=1}^{n} c_i^2\left(-2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right) - \left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \frac{1}{T}\mathcal{R}_{2iT}\right) \\
& +\frac{1}{n^{3/4}}\sum_{i=1}^{n} c_i^3\left(-2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right) + \frac{1}{T}\mathcal{R}_{3iT}\right) \\
& +\frac{1}{n}\sum_{i=1}^{n} c_i^4\left(-\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right)^2 + \frac{1}{T}\mathcal{R}_{4iT}\right),
\end{aligned}$$

where $\frac{1}{n}\sum_{i=1}^{n}\mathcal{R}_{kiT} = O_p(1)$ for $k = 2, ..., 4$.

Computation of $V_{fe222,nT}(\mathbb{C})$:
From a direct calculation we have

$$\begin{aligned}
& \left(\frac{1}{T}\Delta\tilde{G}'\Delta\tilde{G}\right)^{-1} - \left(\frac{1}{T}\Delta_{c_i}\tilde{G}'\Delta_{c_i}\tilde{G}\right)^{-1} \\
= & \begin{pmatrix} 0 & \frac{1}{\sqrt{T}}\left(\frac{c_i}{n^{1/4}} - \frac{1}{2}\frac{c_i^2}{n^{1/2}} + \frac{1}{6}\frac{c_i^3}{n^{3/4}}\right) \\ \frac{1}{\sqrt{T}}\left(\frac{c_i}{n^{1/4}} - \frac{1}{2}\frac{c_i^2}{n^{1/2}} + \frac{1}{6}\frac{c_i^3}{n^{3/4}}\right) & \frac{c_i}{n^{1/4}}\left(2\left(\frac{1}{T}\sum_{t=1}^{T}\frac{t}{T}\right)\right) - \frac{2}{3}\frac{c_i^2}{n^{1/2}} + \frac{1}{3}\frac{c_i^3}{n^{3/4}} - \frac{1}{9}\frac{c_i^4}{n} \end{pmatrix} \\
& +O\left(\frac{1}{n^{1/2}T}\right),
\end{aligned}$$

where $O\left(\frac{1}{n^{1/2}T}\right)$ holds uniformly across i because the support of $c_i's$ is bounded.
Then,

$$\begin{aligned}
& V_{fe222,nT}(\mathbb{C}) \\
= & \frac{1}{n^{1/4}}\sum_{i=1}^{n} c_i\left(\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 - \left(\frac{y_{i0}}{\sqrt{T}}\right)^2\right) \\
& +\frac{1}{n^{1/2}}\sum_{i=1}^{n} c_i^2\left(\frac{4}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \frac{1}{3}\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}-y_{i0}}{\sqrt{T}}\right)\right) \\
& +\frac{1}{n^{3/4}}\sum_{i=1}^{n} c_i^3\left(2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right)\right) \\
& +\frac{1}{n}\sum_{i=1}^{n} c_i^4\left(\frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right) - \frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2\right) + o_p(1),
\end{aligned}$$

where the $o_p(1)$ error holds as $(n, T\to\infty)$ with $\frac{n^{3/4}}{T}\to 0$.

Putting the terms in $V_{fe21,nT}(\mathbb{C})$, $V_{fe221,nT}(\mathbb{C})$, and $V_{fe222,nT}(\mathbb{C})$ together, we have the required result. ■

Proof of Lemma 12

Part (a). First, notice from

$$y_{it}^2 - y_{it-1}^2 = \left(\rho_i^2 - 1\right) y_{it-1}^2 + 2\rho_i y_{it-1} u_{it} + u_{it}^2 \text{ for } t \geq 1,$$

we have

$$\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 - \left(\frac{y_{i0}}{\sqrt{T}}\right)^2 = \left(\rho_i^2 - 1\right) \frac{1}{T} \sum_{t=1}^{T} y_{it-1}^2 + 2\rho_i \frac{1}{T} \sum_{t=1}^{T} y_{it-1} u_{it} + \frac{1}{T} \sum_{t=1}^{T} u_{it}^2.$$

From $\Delta y_{it} = (\rho_i - 1) y_{it-1} + u_{it}$, we have

$$2\frac{1}{T} \sum_{t=1}^{T} \Delta y_{iy} y_{it-1} = 2(\rho_i - 1) \frac{1}{T} \sum_{t=1}^{T} y_{it-1}^2 + \frac{1}{T} \sum_{t=1}^{T} y_{it-1} u_{it}.$$

Then,

$$\begin{aligned}
& \frac{1}{n^{1/4}\hat{\sigma}^2} \sum_{i=1}^{n} c_i \left[\frac{2}{T} \sum_{t=1}^{T} \Delta y_{it} y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \left(\frac{y_{i0}}{\sqrt{T}}\right)^2 + \hat{\sigma}^2\right] \\
= \; & \frac{1}{n^{1/4}\hat{\sigma}^2} \sum_{i=1}^{n} c_i \left[-(\rho_i - 1)^2 \frac{1}{T} \sum_{t=1}^{T} y_{it-1}^2 + 2(1 - \rho_i) \frac{1}{T} \sum_{t=1}^{T} y_{it-1} u_{it} - \left(\frac{1}{T} \sum_{t=1}^{T} u_{it}^2 - \hat{\sigma}^2\right)\right] \\
= \; & -\frac{1}{n^{1/4}\hat{\sigma}^2} \sum_{i=1}^{n} c_i \left(\frac{1}{T} \sum_{t=1}^{T} u_{it}^2 - \hat{\sigma}^2\right) + O_p\left(\frac{n^{1/4}}{T}\right) + O_p\left(\frac{1}{T}\right),
\end{aligned}$$

where the last line holds because

$$\frac{1}{n^{1/4}\hat{\sigma}^2} \sum_{i=1}^{n} c_i (\rho_i - 1)^2 \frac{1}{T} \sum_{t=1}^{T} y_{it-1}^2 = \frac{n^{1/4}}{T\hat{\sigma}^2} \left(\frac{1}{n} \sum_{i=1}^{n} c_i \theta_i^2 \left(\frac{1}{T^2} \sum_{t=1}^{T} y_{it-1}^2\right)\right) = O_p\left(\frac{n^{1/4}}{T}\right),$$

and

$$\frac{1}{n^{1/4}\hat{\sigma}^2} \sum_{i=1}^{n} c_i (1 - \rho_i) \frac{1}{T} \sum_{t=1}^{T} y_{it-1} u_{it} = \frac{1}{T} \frac{2}{n^{1/2}\hat{\sigma}^2} \sum_{i=1}^{n} c_i \left(\frac{1}{T} \sum_{t=1}^{T} y_{it-1} u_{it}\right) = O_p\left(\frac{1}{T}\right).$$

Notice that

$$\begin{aligned}
& \frac{1}{n^{1/4}\hat{\sigma}^2} \sum_{i=1}^{n} c_i \left(\frac{1}{T} \sum_{t=1}^{T} u_{it}^2 - \hat{\sigma}^2\right) \\
= \; & \frac{1}{n^{1/4}\sigma^2} \sum_{i=1}^{n} c_i \left(\frac{1}{T} \sum_{t=1}^{T} u_{it}^2 - \sigma^2\right) + \left(\frac{1}{\hat{\sigma}^2} - \frac{1}{\sigma^2}\right) \frac{1}{n^{1/4}} \sum_{i=1}^{n} c_i \frac{1}{T} \sum_{t=1}^{T} u_{it}^2 \\
= \; & O_p\left(\frac{n^{1/4}}{T^{1/2}}\right) + O_p\left(\max\left\{\frac{1}{n^{1/2}T^{1/2}}, \frac{1}{T}\right\}\right) O_p\left(n^{3/4}\right) = o_p(1),
\end{aligned}$$

where the second equality holds because $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(u_{it}^{2}-\sigma^{2}\right)=O_{p}\left(1\right)$, $\frac{1}{n}\sum_{i=1}^{n}c_{i}\frac{1}{T}\sum_{t=1}^{T}u_{it}^{2}=O_{p}\left(1\right)$ and by Assumption 4 and the last equality holds because $\frac{n^{3/4}}{T}\to 0$ (Assumption 3). Therefore, we have all the required result for Part (a). ■

Part (b).

By Lemma 17(a) and Assumptions 4 and 3, we have

$$\frac{1}{n^{1/2}\hat{\sigma}^{2}}\sum_{i=1}^{n}c_{i}^{2}\left[\begin{array}{c}\left(\frac{1}{T^{2}}\sum_{t=1}^{T}y_{it-1}^{2}-\hat{\sigma}^{2}\frac{1}{T}\sum_{t=1}^{T}\frac{t-1}{T}\right)+\frac{1}{3}\left\{\left(\frac{y_{iT}}{\sqrt{T}}\right)^{2}-\hat{\sigma}^{2}\right\}\\ -\left\{2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=2}^{T}\frac{t-1}{T}y_{it-1}\right)-\hat{\sigma}^{2}\frac{2}{T}\sum_{t=1}^{T}\left(\frac{t-1}{T}\right)^{2}\right\}\end{array}\right]$$
$$\Rightarrow\quad N\left(-\frac{1}{90}E\left(c_{i}^{2}\theta_{i}^{2}\right),\frac{1}{45}E\left(c_{i}^{4}\right)\right).$$

Then, for the required result for Part (b), it remains to show that

$$\frac{1}{n^{1/2}\hat{\sigma}^{2}}\sum_{i=1}^{n}c_{i}^{2}\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}-y_{i0}}{\sqrt{T}}\right)=o_{p}\left(1\right),$$

which follows because

$$\begin{aligned}
&\frac{1}{n^{1/2}\hat{\sigma}^{2}}\sum_{i=1}^{n}c_{i}^{2}\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}-y_{i0}}{\sqrt{T}}\right)\\
=\;&\frac{1}{n^{1/2}\hat{\sigma}^{2}}\sum_{i=1}^{n}c_{i}^{2}\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}\left(0\right)-y_{i0}}{\sqrt{T}}\right)+\frac{1}{n^{1/2}\hat{\sigma}^{2}}\sum_{i=1}^{n}c_{i}^{2}\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}-y_{iT}\left(0\right)}{\sqrt{T}}\right)\\
=\;&O_{p}\left(\frac{1}{\sqrt{T}}\right)+O_{p}\left(\frac{n^{1/4}}{T^{1/2}}\right),
\end{aligned}$$

where the last line holds because

$$\frac{1}{n^{1/2}\hat{\sigma}^{2}}\sum_{i=1}^{n}c_{i}^{2}\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}\left(0\right)-y_{i0}}{\sqrt{T}}\right)=\frac{1}{\sqrt{T}}\left(\frac{1}{n^{1/2}\hat{\sigma}^{2}}\sum_{i=1}^{n}c_{i}^{2}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}\right)\right)=\frac{1}{\sqrt{T}}O_{p}\left(1\right),$$

and by (29),

$$\begin{aligned}
&\frac{1}{n^{1/2}\hat{\sigma}^{2}}\sum_{i=1}^{n}c_{i}^{2}\left(\frac{y_{i0}}{\sqrt{T}}\right)\left(\frac{y_{iT}-y_{iT}\left(0\right)}{\sqrt{T}}\right)\\
=\;&-\left(\frac{n^{1/4}}{T^{1/2}}\right)\frac{1}{\hat{\sigma}^{2}}\left(\frac{1}{n}\sum_{i=1}^{n}c_{i}^{2}\theta_{i}y_{i0}\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T-1}u_{ip}\right)+o_{p}\left(1\right)\right)=\frac{n^{1/4}}{T^{1/2}}O_{p}\left(1\right).\ ■
\end{aligned}$$

Part (c).

Under Assumption 4, we have

$$\begin{aligned} & \frac{1}{n\hat{\sigma}^2}\sum_{i=1}^{n}c_i^4\left[-\left(\frac{1}{T\sqrt{T}}\sum_{t=2}^{T}\frac{t-1}{T}y_{it-1}\right)^2+\frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=2}^{T}\frac{t-1}{T}y_{it-1}\right)-\frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2\right] \\ = & \frac{1}{n\sigma^2}\sum_{i=1}^{n}c_i^4\left[-\left(\frac{1}{T\sqrt{T}}\sum_{t=2}^{T}\frac{t-1}{T}y_{it-1}\right)^2+\frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=2}^{T}\frac{t-1}{T}y_{it-1}\right)-\frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2\right] \\ & +o_p(1), \end{aligned}$$

and the required result for Part (c) follows by the WLLN (*e.g.* Corollary 1 in Phillips and Moon (1999)). ■

Proof of Lemma 15

Lemma 15 holds by Lemma 17(a) with $c_i = 1$ and Assumption 4. ■

Proof of Lemma 16

Notice that we can decompose $\sqrt{n}\left(\frac{1}{nT^2\hat{\sigma}^2}tr\left(ZQ_GZ'\right)-\omega_{2T}\right)$ as

$$V_{o,nT}=\sqrt{n}\left(\frac{1}{nT^2\sigma^2}tr\left(YQ_GY'\right)-\omega_{2T}\right)+\frac{tr\left(YQ_GY'\right)}{nT^2}\sqrt{n}\left(\frac{1}{\hat{\sigma}^2}-\frac{1}{\sigma^2}\right).$$

Then, lemma 16 holds by Lemma 17(b) and Assumption 4. ■

References

[1] Choi, I. (2001): Unit Root Tests for Panel Data, *Journal of International Money and Finance*, 20, 249–272.

[2] Dufour, J. and M. King (1991): Optimal Invariant Tests for the Autocorrelation Coefficient in Linear Regressions with Stationary or Nonstationary AR(1) Errors, *Journal of Econometrics*, 47, 115–143.

[3] Elliott, G. T. Rothenberg, and J. Stock (1996): Efficient Tests for an Autoregressive Unit Root, *Econometrica*, 64, 813–836.

[4] Im, K., H. Pesaran, Y. Shin (1997): Testing for Unit Roots in Heterogeneous Panels, *Mimeo.*

[5] King, M. (1988): Towards a Theory of Point Optimal Testing, *Econometric Reviews*, 6, 169–218.

[6] Lehmann, E. (1959): *Testing Statistical Hypotheses*, New York, Wiley.

[7] Levin, A., F. Lin, and C. Chu (2002): Unit Root Tests in Panel Data: Asymptotic and Finite-Sample Properties, *Journal of Econometrics*, 108, 1–24.

[8] Maddala, G.S. and S. Wu (1999): A Comparative Study of Unit Root Tests with Panel Data and a New Simple Test, *Oxford Bulletin of Economics and Statistics*, 61, 631–651.

[9] Moon, H.R. and B. Perron (2003a): Testing for a Unit Root in Panels with Dynamic Factors, CLEO Working Paper, USC.

[10] Moon, H.R. and B. Perron (2003b): Asymptotic Local Power of a Test for Unit Roots in Panels with Fixed Effects, Mimeo.

[11] Moon, H.R. and P.C.B. Phillips (2002): GMM Estimation of Autoregressive Roots Near Unity with Panel Data, CLEO Working Paper, USC.

[12] Phillips, P. C. B. (2001). New Unit Root Asymptotics in the Presence of Deterministic Trends.. *Journal of Econometrics*, 2001, Vol. 11, pp. 323-353.

[13] Phillips, P.C.B. and H.R. Moon (1999): Linear Regression Limit Theory for Nonstationary Panel Data, *Econometrica*, 67, 1057–1111.

[14] Phillips, P. C. B. and D. Sul (2003). Dynamic Panel Estimation and Homogeneity Testing under Cross Section Dependence. *Econometrics Journal.*

[15] Ploberger, W. and P.C.B. Phillips (2002): Optimal Testing for Unit Roots in Panel Data, *Mimeo.*

[16] Quah, D. (1994): Exploiting Cross-Section Variations for Unit Root Inference in Dynamic Panels, *Economics Letters*, 44, 9–19.

Table 1. Size of tests
DGP: $z_{it} = \alpha_{i0} + \alpha_{i1}t + z_{it}^0$
$z_{it}^0 = z_{it-1}^0 + e_{it}$
$\alpha_{i0}, \alpha_{i1}, e_{it} \sim iidN(0,1)$

n	T	$c_i = 1$	$c_i = 2$	$c_i = 0.5$	Ploberger-Phillips	Moon-Phillips	t test
10	100	12.3	1.4	26.7	1.3	0.8	6.1
20	100	15.9	4.3	36.7	1.2	2.0	7.6
30	100	22.2	7.3	42.7	3.1	0.7	8.1
10	300	2.9	0.5	8.7	1.2	2.1	4.9
20	300	6.3	1.2	11.3	2.4	2.3	5.3
30	300	7.5	2.8	14.0	2.6	2.6	5.8
10	500	2.8	0.1	5.0	2.2	2.5	5.0
20	500	4.7	0.8	6.6	2.4	2.9	5.2
30	500	5.1	1.3	10.1	2.6	3.3	5.4

Table 2. Size-adjusted power of tests
DGP: $z_{it} = \alpha_{i0} + \alpha_{i1}t + z_{it}^0$
$z_{it}^0 = \rho_i z_{it-1}^0 + e_{it}$
$\alpha_{i0}, \alpha_{i1}, e_{it} \sim iidN(0,1)$
$\rho_i \sim U[0.98, 1]$

n	T	$c_i = 1$	$c_i = 2$	$c_i = 0.5$	Ploberger-Phillips	Moon-Phillips	t test
10	100	7.3	7.6	6.5	7.6	7.3	4.9
20	100	12.4	10.2	6.7	11.5	6.8	4.0
30	100	11.3	10.0	8.2	9.4	13.6	2.9
10	300	27.7	28.4	24.4	27.0	22.5	2.3
20	300	45.5	46.9	53.2	49.9	39.3	2.5
30	300	58.8	57.9	59.9	64.5	52.3	1.8
10	500	56.6	56.8	58.6	57.7	53.7	0.9
20	500	81.5	87.3	84.5	83.1	76.9	0.5
30	500	93.4	92.4	91.8	94.0	92.1	0.4