Problems and Solutions

Problems

97.1.1 *Standard Errors for the Long-Run Variance Matrix*, proposed by Paolo Paruolo. Consider a p-dimensional vector autoregressive model of order $k > 1$parameterized in ECM form as follows

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \epsilon_t \tag{1}$$

where $\epsilon_t \sim N(0, \Omega)$, Ω positive definite ; see Johansen(1995,Ch.2), to which we refer for general definitions and background. Let $\Gamma = I - \sum_{i=1}^{k-1} \Gamma_i$ and assume that I(1) conditions are satisfied (Johansen, 1995, Theorem 4.2) such that, by an appropriate choice of initial values, X_t has the representation

$$X_t = C \sum_{i=1}^{t} \epsilon_i + Y_i \tag{2}$$

where Y_i is stationary and $C = \beta_\perp(\alpha'_\perp \Gamma \beta_\perp)^{-1}\alpha'_\perp$. Let $\widehat{}$ indicate maximum likelihood estimates, $P_a = a(a'a)^{-1}a'$ the projection matrix onto *span(a)*, and vec$(\cdot)$ the column stacking operator.

1. The long-run variance of the system is given by $C\Omega C'$ (see Johansen (1995, p.49). Show that

$$T^{1/2} vec(\widehat{C}\widehat{\Omega}\widehat{C}' - C\Omega C') \xrightarrow{w} N(0, 2P_D(C\Omega(\Omega^{-1} + 2\xi'\Sigma^{-1}\xi)\Omega C' \otimes C\Omega C')P_D), \tag{3}$$

   where D is the duplication matrix of order p (see Magnus and Neudecker, 1988, Sect, 3.8) and where $\Sigma = Var(Z_t)$, $Z_t = (X'_{t-1}\beta, \Delta X'_{t-1}, \cdots, \Delta X'_{t-k})'$, and $\xi' = ((C'\Gamma - I)\alpha(\alpha'\alpha)^{-1}, C', \cdots, C')$, following the notation in Johansen(1995, Theorem 13.7).onto the
2. Consider the orthogonalized shocks

$$\eta_t = A^{-1}\varepsilon_t,$$

   where $\Omega = AA'$ is the Choleski decomposition of Ω. By interpreting η_t as structural innovations, (2) can be rewritten as

$$X_t = C^* \sum_{i=1}^{t} \eta_i + Y_i \tag{4}$$

   where $C^* = CA$ represents the loadings of the variables in X_t onto the common structural trends. Show that

$$T^{1/2} vec(\widehat{C} - C^*) \xrightarrow{w} N(0, (A'\xi'\Sigma^{-1}\xi A \otimes C\Omega C') + \frac{1}{2}B(\Omega \otimes \Omega)B'), \tag{5}$$

where $B = (I \otimes C)D(D'(A \otimes I)D)^{-1}D'$.

3. Discuss the singularity of the asymptotic covariance matrices in 3 and 5.
4. Discuss the effects of introducing a drift μ in the right-hand side of 1 on the preceding results.

Hint : Use Theorem 13.7 from Johansen(1995) or Theorem 7.1 from Paruolo (1997). Note that Johansen uses implicitly a *row* stacking operator.

References

[1] Johansen, S. (1995) *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models*, Oxford: Oxford University Press.

[2] Magnus, J.R. and H. Neudecker (1988) *Matrix Differential Caculus with Applications in Statistics and Econometrics*. Chichester, UK: John Wiley & Sons.

[3] Paruolo, P (1997) “Asymptotic inference on the moving average impact matrix in cointegrated I(1) VAR systems,” *Econometric Theory* 13, 79-11.

Solutions

96.1.1. *Approximating the finite Sample Bias for Maximum Likelihood Estimators Using the Score* — Solution, proposed by Bert Lambrecht, William Perraudin, and Stephen Satchell. Define $L(\beta)$ as the logarithm of the likelihood function of the data divided by n, wherer n is the sample size. Assume that the usual regularity condotions for parameter differentiation hold. Define $L_i(\beta)$ to the i-th derivative of L with respect of β and assume that these derivatives exist. Let $\lambda_i(\beta) = E(L_i(\beta))$, where $L_i(\beta)$ and $\lambda_i(\beta)$ are evaluated at the true parameter value. Define

$$l_1 = \sqrt{n}L_1(\beta) \tag{6}$$
$$l_2 = \sqrt{n}(L_2(\beta) - \lambda_2(\beta)) \tag{7}$$
$$l_3 = \sqrt{n}(L_3(\beta) - \lambda_3(\beta)) \tag{8}$$

where $l_{1,}l_{2,}l_3$ s $O_p(1)$.